Grade 10 Mathematics

Learners' Book

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Back Matter

Chapter 1

Numbers and Algebra

1.1 Real Numbers

Real numbers represented as $\mathbb{R}$ are a set of numbers that include all rational and irrational numbers. They can be positive, negative, or zero, and can be represented on the number line. The table below shows classification of real numbers with examples.

Category	**Examples**
Whole numbers ($\mathbb{W}$)	$0, 1, 2, 3, 4, ...$
Integers($\mathbb{Z}$)	$-4, -3, -2, 0, 1, 2, ...$
Fractions	$\frac{1}{2}, \frac{2}{3}, \frac{1}{3}, \frac{5}{4}, \frac{12}{6}, ...$
Decimals	$0.2, 4.5, -2.6, -3.8, ...$
Irrational numbers	$\sqrt{2}, \sqrt{7}, \pi, ...$

Note 1.1.1 Real numbers do not include complex numbers like the square root of negative one $\left(\sqrt{-1}\right)$.

How do we use real numbers in day-to-day activities?

We use real numbers in everyday life for tasks such as managing finances while budgeting or shopping, measuring ingredients for cooking, tracking time and distance when traveling, and interpreting data on digital devices.

1.1.1 Classifying whole numbers as odd, even, prime and composite in different situations

Whole numbers can be classified as odd, even, prime or composite based on their divisibility and factor properties.

Activity 1.1.1

1. Working in groups, write numbers from 1 to 100.
2. Recall: What is an odd number? What is an even number?
3. Sort the numbers you wrote down into odd or even.
4. What patterns do you notice between even and odd numbers?
5. Identify prime and composite numbers from the list of even and odd numbers you have.
6. Discuss how a number is classified.

7. Identify prime numbers that are even.
8. Brainstorm a real life example where you can find odd, even, prime and composite numbers e.g pairs of shoes are even numbers.
9. Describe why classifying numbers in real-life could be useful.
10. Share your work with your fellow learners.

Even and Odd numbers.

Key Takeaway 1.1.2 **Even numbers** are numbers that are divisible by 2. **Odd numbers** are numbers that when divided by 2, you get a remainder.
Properties

- The sum or difference of two even numbers is even.
  Example: $6 + 4 = 10$ and $6 - 2 = 4$.
- The sum or difference of two odd numbers is even.
  Example: $7 + 3 = 10$ and $11 - 9 = 2$
- The sum or difference of an even and an odd number is always odd.
  Example: $8 + 5 = 13$ and $8 - 5 = 3$
- When two odd integers are multiplied, the result is always an odd number.
  Example: $3 \times 3 = 9$
- When two even integers are multiplied, the result is an even number.
  Example: $12 \times 12 = 144$
- An even number multiplied by an odd number equals an even number.
  Example: $12 \times 3 = 36$

Remark 1.1.3 We can only talk about even and odd numbers if they are integers (i.e., elements of $\mathbb{Z}$).

Criterion 1: How to Identify Even or Odd Integers
A number is even if it ends with one of the digits: $0, 2, 4, 6, 8$.
A number is odd if it ends with one of the digits: $1, 3, 5, 7, 9$.

Note 1.1.4 This method works only for integers (whole numbers, including zero and negatives).

Example 1.1.5 Classify the following numbers as even or odd:

a) 1107
b) 2028
c) 3333
d) 5052
e) 1800
f) 1349

Solution.

a) 1107 is an odd number since its last digit is 7.
b) 2028 is an even number since its last digit is 8.
c) 3333 is an odd number since its last digit is 3.
d) 5052 is an even number since its last digit is 2 .
e) 1800 is an even number since its last digit is 0.
f) 1349 is an odd number since it ends with 9.

□

Example 1.1.6 Kirui has 35 cows on his farm and wants to group them into 2 pens. Will each pen have an equal number of cows? Explain using properties of even and odd numbers.

Solution. First we identify whether 35 is even or odd.

Since the last digit is 5, this makes 35 an odd number.

Since an odd number cannot be divided evenly into equal groups, the cows can only be divided into two groups: one with 18 cows and the other with 17 cows. Therefore, the cows cannot be shared evenly across all pens. ☐

Exercise

1. Classify the following numbers as even or odd using the given criterion.

a) 1008
b) 1521
c) 2117
d) 625
e) 1314
f) 1703
g) 9272
h) 22801

Answer.

a) 1008 is an even number since its last digit is 8.
b) 1521 is an odd number since its last digit is 1.
c) 2117 is an odd number since its last digit is 7.
d) 625 is an odd number since its last digit is 5.
e) 1314 is an even number since its last digit is 4.
f) 1703 is an odd number since its last digit is 3.
g) 9272 is an even number since its last digit is 2.
h) 22801 is an odd number since its last digit is 1.

2. List all the numbers between 8102 and 8130 and identify odd and even number from the range.

Answer. Even numbers: $8104, 8106, 8108, 8110, 8112, 8114, 8116, 8118, 8120, 8122, 8124, 8126, 8128$.

Odd numbers: $8103, 8105, 8107, 8109, 8111, 8113, 8115, 8117, 8119, 8121, 8123, 8125, 8127, 8129$.

3. Find the sum of the first 20 numbers and determine if the result is odd or even.

Answer. The sum of the first 20 natural numbers is 210, which is an even number.

4. A grade 10 class has 52 students and their class teacher wanted to group them in pairs. Will each group have an equal number of students? Explain using odd or even properties.

Answer. Since 52 is an even number (it ends with 2), it can be divided evenly into pairs.

Further activity.

Mutula is organizing a party, and he has 35 party hats. Can Mutula arrange the hats in rows where each row has the same number of hats? What does this tell you about the number 35?

Prime and Composite numbers.

Key Takeaway 1.1.7 A prime number is a number that has only two factors, that is, 1 and itself. Example:$2, 3, 5, 7, 11, ...$

A factor of a number is a natural number that divides it exactly with no remainder.

For example, 6 is not a prime number because it has more than two factors: $1, 2, 3$, and 6. That is, $6 = 1 \times 6$ and $6 = 2 \times 3$.

To identify if a number is prime, check if the number has only two factors: 1 and itself.

Composite numbers are natural numbers greater than 1 that have more than two factors.

Examples: 4(factors : $1, 2, 4$), 6(factors: $1, 2, 3, 6$), 9(factors: $1, 3, 9$). Hence $4, 6$ and 9 are composite numbers since they have more than 2 factors.

To identify if a number is composite, check if the number has more than two factors. In other words, if it can be divided exactly by numbers other than 1 and itself, then it is a composite number.

Examples:

a. 4 is composite because $4 = 1 \times 4$ and 2×2.

b. 9 is composite because $9 = 1 \times 9$ and 3×3.

c. 10 is composite because $10 = 1 \times 10$ and 2×5

Properties

Every composite number has prime factors. That is, it can be broken down into a product of prime numbers.

- The only even prime number is 2 .
- All other even numbers greater than 2 are composite.
- The smallest composite number is 4.
- Odd composite numbers are odd natural numbers greater than 1 that are not prime (e.g., $9, 15, 21, ...$).

Think of prime numbers as building blocks, and composite numbers as being made by combining those blocks.

Note 1.1.8 0 and 1 are neither prime nor composite.

Example 1.1.9 Which of the following numbers are prime and which are composite?

a) 1021
b) 1111
c) 1999
d) 2000
e) 3011
f) 3500

Solution.

a) 1021 has no factors other than 1 and 1021 itself.

Hence 1021 is a prime number.

b) $1111 = 11 \times 101$ meaning it is divisible by both 11 and 101. it is also divisible by 1 and itself.

Therefore, 1111 is a composite number.

c) 1999 cannot be divided by any number other than 1 and 1999 itself.

This means that 1999 is a prime number.

d) $2000 = 2^4 \times 5^3$ which implies that 2 and 5 are its factors. Also, it is divisible by 1 and itself, 20 and 100, 2 and 1000.

Therefore, 2000 is a composite number

e) 3011 has no factors other than 1 and 3011 itself, making it a prime number.

f) $3500 = 2^2 \times 5^3 \times 7$ meaning 3500 has other factors other than 1 and 3500.

Hence 3500 is a composite number.

□

Checkpoint 1.1.10 Classifying Numbers as Even or Odd. Load the question by clicking the button below.Classify the following numbers as even or odd.

1. 35496: [Even | Odd]
2. 16853: [Even | Odd]
3. 44631: [Even | Odd]
4. 41961: [Even | Odd]
5. 25759: [Even | Odd]
6. 35687: [Even | Odd]
7. 36122: [Even | Odd]
8. 25750: [Even | Odd]

Answer 1. Even
Answer 2. Odd
Answer 3. Odd
Answer 4. Odd
Answer 5. Odd
Answer 6. Odd
Answer 7. Even
Answer 8. Even

Solution. Worked Solution A number is ***even*** if it is divisible by 2 without leaving a remainder. A number is ***odd*** if it is not divisible by 2 such that when the number is divided by 2, the remainder = 1

Number	Calculation	Result
35496	$35496 \div 2 = 17748$ remainder 0	***Even***
16853	$16853 \div 2 = 8426$ remainder 1	***Odd***
44631	$44631 \div 2 = 22315$ remainder 1	***Odd***
41961	$41961 \div 2 = 20980$ remainder 1	***Odd***
25759	$25759 \div 2 = 12879$ remainder 1	***Odd***
35687	$35687 \div 2 = 17843$ remainder 1	***Odd***
36122	$36122 \div 2 = 18061$ remainder 0	***Even***
25750	$25750 \div 2 = 12875$ remainder 0	***Even***

You can also classify a number as even if its last digit is 0, 2, 4, 6, or 8 and classify as odd if its last digit is 1, 3, 5, 7, or 9

Checkpoint 1.1.11 Classifying a Number as Odd, Even, Prime or Composite. Load the question by clicking the button below.During a mathematics lesson, Grade 10 students were asked to classify the number 45038.

Which of the following statements are true? Select all that apply.

(1) It is odd

(2) It is prime

(3) It is composite

(4) It is even

Answer. (3)

Solution. ***Worked Solution***

Given the number: 45038***1. Test if the number is even or odd***

We check the last digit of the number.

The last digit is 8, which is even. Therefore, the number is ***even***.***2. Test if the number is prime or composite***

The number has a divisor other than 1 and itself, so it is ***composite***.Therefore, the correct classification of 45038 is:

- It is even
- It is composite

Checkpoint 1.1.12 Classification of Prime and Composite Numbers. Load the question by clicking the button below.Classify each of the following numbers as ***Prime***, ***Composite***, or ***Neither***

1. 1: [Prime | Composite | Neither]
2. 524: [Prime | Composite | Neither]
3. 641: [Prime | Composite | Neither]
4. 449: [Prime | Composite | Neither]
5. 0: [Prime | Composite | Neither]
6. 302: [Prime | Composite | Neither]
7. 557: [Prime | Composite | Neither]
8. 504: [Prime | Composite | Neither]

Answer 1. Neither

Answer 2. Composite

Answer 3. Prime

Answer 4. Prime

Answer 5. Neither

Answer 6. Composite

Answer 7. Prime

Answer 8. Composite

Solution. Worked Solution

- ***Prime:*** is a number greater than 1 with exactly two distinct positive divisors: 1 and itself.

- ***Composite:*** is a number greater than 1 with more than two positive divisors.
- ***Neither:*** Numbers 0 and 1 are neither prime nor composite.

Number	Check	Classification
1	n < 2	***Neither***
524	Has factors other than 1 and itself	***Composite***
641	No divisors except 1 and itself	***Prime***
449	No divisors except 1 and itself	***Prime***
0	n < 2	***Neither***
302	Has factors other than 1 and itself	***Composite***
557	No divisors except 1 and itself	***Prime***
504	Has factors other than 1 and itself	***Composite***

Note that:

- If a number > 2 ends in 0, 2, 4, 6, or 8, it's composite.
- If a number > 5 ends in 0 or 5, it's composite.
- If the sum of digits is divisible by 3, the number is composite.
- 0 and 1 are neither prime nor composite.

Exercises

1. Classify the following numbers as prime or composite.

a) 1004
b) 1009
c) 1031
d) 1050
e) 1061
f) 1033
g) 1049
h) 3092
i) 2046

Answer.

a) 1004 is composite because $1004 = 2 \times 502$.

b) 1009 is prime because it has no factors other than 1 and 1009.

c) 1031 is prime because it has no factors other than 1 and 1031.

d) 1050 is composite because $1050 = 2 \times 525$.

e) 1061 is prime because it has no factors other than 1 and 1061.

f) 1033 is prime because it has no factors other than 1 and 1033.

g) 1049 is prime because it has no factors other than 1 and 1049.

h) 3092 is composite because $3092 = 2 \times 1546$.

i) 2046 is composite because $2046 = 2 \times 1023$.

2. Simplify each of the following expressions and state whether the result is a prime or composite number.

a) $1024 \times 5 \div 4$

b) $\sqrt{144} \times 3 - 9 + 4$

c) $\sqrt{64} \times 5$

d) $4^2 \times 2 + 4$

e) $49^2 + 6 \div 7$

f) $\sqrt{25} \times 2 - 8$

Answer.

a) $1024 \times 5 \div 4 = 1280$, which is composite because 1280 is divisible by numbers other than 1 and itself.

b) $\sqrt{144} \times 3 - 9 + 4 = 31$, which is prime because it has no factors other than 1 and 31.

c) $\sqrt{64} \times 5 = 40$, which is composite.

d) $4^2 \times 2 + 4 = 36$, which is composite.

e) $49^2 + 6 \div 7 = 2401 + \frac{6}{7} = 2401.857$ when given to 3 decimal places, and is classified as neither prime nor composite since it is not a natural number.

f) $\sqrt{25} \times 2 - 8 = 2$, which is prime because it has no factors other than 1 and 2.

3. The number 51 is suspected to be prime. Use divisibility rules to determine whether it is a prime or composite number.

Answer. $51 = 3 \times 17$, which means that 51 has factors other than 1 and itself. Therefore, 51 is a composite number.

4. A teacher writes a two-digit number on the board. The number is prime, less than 50, and ends with 3. List all possible numbers it could be.

Answer. The possible two-digit prime numbers that end with 3 and are less than 50 are: $13, 23, 43$.

5. A marathon is divided into 42-kilometer relay sections. Each runner must cover a distance (in km) that is a composite number. List three possible distances a runner could cover.

Answer. Any three composite numbers less than or equal to 42 km, such as 4 km, 6 km, and 9 km.

6. A class of students forms a rectangular grid. The total number of students is 273. Determine whether this number is prime or composite and explain your reasoning.

Answer. $273 = 13 \times 21$, which means that 273 has factors other than 1 and itself. Therefore, 273 is a composite number.

7. If a number is divisible by 2 and 3, what smallest composite number could it be?

Answer. The smallest composite number divisible by both 2 and 3 is 6 since $6 = 2 \times 3$.

1.1.2 Classifying real numbers as rational and irrational in different situations

Activity 1.1.2

1. Working in groups, choose any set of natural numbers between 1 and 10 (e.g., $2, 3, 5, 7, ..$)

   • Use these numbers to create at least two fractions:

   One proper fraction (e.g., 3/5)

   One improper fraction (e.g., 7/4)

2. Use a calculator to divide each of your fractions.

   Write down the decimal value of each.

   Example: $7 \div 4 = 1.75$.

3. Choose any set of Natural numbers between 1 and 20.

   Write each number as a square root.

   Example: $\sqrt{9} = 3$(a rational number),$\sqrt{2} = 1.414...$(an irrational number).

4. Classify each number you have created (fractions, decimals, and square roots) as either rational or irrational.

5. What do you notice about the decimal form of rational numbers compared to irrational numbers?

   Are there any patterns?

6. Discuss your observations with your fellow learners.

Activity 1.1.3

1. Working in groups, write any 5 numbers between 1 and 10 (e.g., $2, 3, 5, 7, ..$)

2. Form at least 3 fractions using the numbers you formed above. Form both proper and improper fractions e.g.,$\frac{2}{5}, \frac{7}{3}$.

3. Divide each of the fractions you formed to express it as a decimal. Example: $2 \div 5 = 0.4, 7 \div 3 = 2.\dot{3}$.

4. Express the numbers 1 to 10 as square roots. Example: $\sqrt{1}, \sqrt{2}, ..., \sqrt{10}$.

5. Classify the fractions, decimals, and square root numbers you have formed as rational and irrational numbers.

6. What do you notice about the decimal forms of fractions?

7. Discuss your work with fellow learners.

Key Takeaway 1.1.13 • **Rational number**($\mathbb{Q}$) : A rational number is any number that can be written as a fraction $\frac{p}{q}$ where p and q are integers and $q \neq 0$.

Example: $\frac{2}{3}, -3, 4$

• **Irrational number:** An irrational number is any number that cannot be expressed as a fraction of two intergers.

Example: $\sqrt{7}, \sqrt{2}, \pi$

• **Integers** consists of positive whole numbers, negative whole numbers and 0.

• The decimal representation of a rational number either terminates (stops at some point) or repeats (continues but has a repeating pattern).

Example: $0.375, 3.45454545...$

• The decimal representation of an irrational number neither terminates (does not stop) nor repeats (continues without a repeated pattern).

Example: $3.14285714..., 4.298103993...$

• The square root of a perfect square is a rational number.

Example: $\sqrt{16} = 4$ and 4 can be written as $\frac{4}{1}$

• The square root of an imperfect square is an irrational number.

Example: $\sqrt{2} = 1.41421356237$ which is an irrational number.

How to determine if a number is rational or irrational.

1. Rule: Check if the number is an integer or a fraction with an integer as the numerator and the denominator. If it is, then it is a rational number.

   Example: 7 or $\frac{4}{5}$ are both rational numbers because they are either whole numbers or simple fractions.

2. Rule: "If the number is in decimal form, check if the decimal stops at some point. If it stops then the number is a rational number."

   Example: 3.25 is rational because the decimal terminates.

3. Rule: If the number is in decimal form, check if the decimal continues. If it continues with a repeated pattern then the number is a rational number and if it continues without a pattern then it is irrational.

   Example (rational): 0.666…(with pattern)

   Example (irrational): 0.1010010001…(no pattern)

4. Rule: "If the number is expressed as a square root, find the square root of the number first and identify if it is a perfect or an imperfect square. If it is a perfect square (results to a whole number) then it is rational and if it is an imperfect square, then it is irrational.

   Example (rational): $\sqrt{49} = 7$

   Example (irrational): $\sqrt{2}$ 1.414213...

Example 1.1.14 Identify if the following numbers are rational or irrational.

a) π

b) $\dfrac{2}{3}$

c) 3.75

d) $\sqrt{20}$

e) $\dfrac{\sqrt{9}}{\sqrt{16}}$

Solution.

1. Since π is defined as 3.1415926.

   We check if the decimal has a pattern.

   Hence π is irrational because its decimal continues without having a repeated pattern.

2. Since $\frac{2}{3}$ is a fraction, we check if the fraction consist of integers with the denominator not equal to zero.

   Therefore $\frac{2}{3}$ is a rational number.

3. To identify if 3.75 is rational or irrational, we check the decimal.

   A rational number can have a terminating decimal or a repeating decimal. In this case, the decimal terminates, so the number is rational

4. For $\sqrt{20}$, first we find its value.

   $\sqrt{20} = 4.472135955$

   $\sqrt{20}$ has decimal which continues without a repeated pattern.

   Hence $\sqrt{20}$ is irrational.

5. The value of $\frac{\sqrt{9}}{\sqrt{16}} = \frac{3}{4}$

   $\frac{3}{4}$ is a fraction with integers on the numerator and the denominator.

   Therefore, $\frac{\sqrt{9}}{\sqrt{16}}$ is rational.

□

Example 1.1.15 Joy is designing a square garden. She measures the total area of the garden to be 50 square meters and wants to find the length of one side. What is the exact length of one side of the garden? Classify the answer as a rational or irrational number.

Solution. Area of the garden = 50 m^2

To find the length of one side of the square, we take the square root of its area.

Side length = $\sqrt{50}$.

Simplifying the side length $\sqrt{50} = \sqrt{(25 \times 5)} = \sqrt{25} \times \sqrt{2} = 5\sqrt{2}$.

Since $\sqrt{2}$ an irrational number, multiplying it by 5 still gives an irrational number.

Therefore, the exact length of one side of the garden is $5\sqrt{2}$ meters, which is an irrational number. □

Checkpoint 1.1.16 Classifying Given Numbers as Irrational. Load the question by clicking the button below.Which of the following numbers are irrational? Select all that apply.

(1) 0.121221222

(2) 5

(3) $\sqrt{5}$

(4) -9

(5) $2^{\frac{3}{2}}$

(6) 5

Answer. $(1), (3), (5)$

Solution. Worked Solution

An irrational number cannot be expressed as a fraction $\frac{p}{q}$ where p and q are integers.

Note that:

- $\sqrt{n}$ is irrational when n is not a perfect square
- π and e are irrational
- Decimals that neither terminate nor repeat are irrational

Therefore, irrational numbers in this list are: $\sqrt{5}, 2^{\frac{3}{2}}, 0.121221222$

Checkpoint 1.1.17 Classifying Rational Numbers from a List. Load the question by clicking the button below.Grade 10 maths teacher has a magical number sorter that only accepts rational numbers. The following numbers are ready to be sorted. Which of them will be accepted? Select all rational numbers.

(1) -9

(2) $\sqrt{5}$

(3) $3\sqrt{7}$

(4) 5

(5) 0.121221222

(6) $\frac{12}{5}$

Answer. $(1), (4), (6)$

Solution. Worked Solution A rational number is any number that can be expressed as a fraction $\frac{p}{q}$ where p and q are integers and $q \neq 0$.

The decimal representation of a rational number either terminates (stops at some point) or repeats (continues but has a repeating pattern).

The square root of a perfect square is a rational number.

Therefore, from the given numbers, the list of rational numbers are: $(-9,\frac{12}{5},5)$

Exercise

1. Classify the following numbers as rational or irrational giving reasons.

a) $\sqrt{25}$

b) 2π

c) $\sqrt{2}$

d) $\frac{7}{3}$

e) 0.25

f) 0.121221222

g) -4.5

h) $\sqrt{2} + \sqrt{8}$

i) $\pi - 3$

j) $\frac{\sqrt{4}}{\sqrt{9}} \times 4$

k) $2 \times \sqrt{2}$

Answer.

(a) $\sqrt{25}$ is rational because it is a perfect square.

(b) 2π is irrational because its a product of a rational number and an irrational number.

(c) $\sqrt{2}$ is irrational because it is an imperfect square.

(d) $\frac{7}{3}$ is rational because it is a fraction with integers on the numerator and denominator.

(e) 0.25 is rational because the decimal terminates.

(f) $0.121221222...$ is irrational because the decimal continues without a repeated pattern.

(g) -4.5 is rational because the decimal terminates.

(h) $\sqrt{2}+\sqrt{8}$ is irrational because it is the sum of two irrational numbers.

(i) $\pi - 3$ is irrational because it is the difference between an irrational number and a rational number.

(j) $\frac{\sqrt{4}}{\sqrt{9}} \times 4 = \frac{8}{3}$ is rational because it is a fraction with integers on the numerator and denominator.

(k) $2 \times \sqrt{2}$ is irrational because it is the product of a rational number and an irrational number.

2. A square garden has a perimeter of 8 units. Find its area and identify if it is a rational or irrational number.

Answer. The area of the square garden is 4 units^2, which is a rational number.

3. For each of the following values of m state whether $\frac{m}{16}$ is rational or irrational.

a) 1

b) -10

c) $\sqrt{2}$

d) $\sqrt{25}$

e) π

Answer.

(a) For $m = 1$, $\frac{1}{16}$ is rational.

(b) For $m = -10$, $\frac{-10}{16} = -\frac{5}{8}$ is rational.

(c) For $m = \sqrt{2}$, $\frac{\sqrt{2}}{16}$ is irrational.

(d) For $m = \sqrt{25}$, $\frac{\sqrt{25}}{16} = \frac{5}{16}$ is rational.

(e) For $m = \pi$, $\frac{\pi}{16}$ is irrational.

4. A car is moving at $\sqrt{225}$ km/h. Is the speed rational or irrational? Explain your answer.

Answer. Since $\sqrt{225} = 15$, which is a whole number, the speed is rational.

5. Iregi a grade 10 student measures a triangular shelf in their home and found out its sides of length was $\sqrt{12}$ meters, $\sqrt{27}$ meters and 5 meters. He wants to find the perimeter of the triangle and identify if it is rational or irrational. Help Iregi to find out if the perimeter is rational or irrational explaining your workings.

Answer. Perimeter $= \sqrt{12} + \sqrt{27} + 5 = 13.660254037844...$ which is irrational because the decimal continues without a repeated pattern.

6. A rectangular garden has a length of 4 meters and a width of $\sqrt{8}$ meters. Find the area of the garden and identify if it is rational or irrational.

Answer. Area $= 4 \times \sqrt{8} = 11.313708498984...$ which is irrational because the decimal continues without a repeated pattern.

1.1.3 Finding reciprocals of real numbers using division

Activity 1.1.4

1 In groups of 5, write down numbers from 2 to 15 in ascendinding order $(2, ..., 15)$.Label this as the first list.

2 Rearrange the numbers from 15 to 2 in descending order $(15, 14, 13, ..., 2)$. Label this as the Second List.

3 In turns, each group member to create a fraction by:

i) Using the first number from the First List as the numerator.

ii) Using the first number from the Second List as the denominator.

4 Find the reciprocals of the fractions you have formed using division.

For example: Fractions formed are $\frac{2}{15}, \frac{3}{14}, \frac{5}{12}, \frac{10}{7}, \frac{7}{10}$ then the reciprocal of $\frac{2}{15}$ using division is $15 \div 2$

5 From the whole numbers 2 to 15 you wrote down, pick any 3 numbers and find its reciprocal using division.

6 Discuss your work with fellow learners.

Key Takeaway 1.1.18 The reciprocal of a number is the number that, when multiplied by the original number, gives a product of 1.

In other words the reciprocal of a real number x is $\frac{1}{x}$, except when $x = 0$ since division by zero is undefined.

For example:

1. The reciprocal of 2 is $\frac{1}{2}$. Using division, the reciprocal of 2 can be found as $1 \div 2$

2. To find the reciprocal of 0.25 using division, you divide 1 by 0.25 i.e $1 \div 0.25$.

3. The reciprocal of $\frac{3}{5}$, using division, is $1 \div \frac{3}{5}$.

To find the reciprocal of a real number using division, follow these steps:

1. Understand the Reciprocal: The reciprocal of a real number x is $\frac{1}{x}$. This means that when you multiply a number by its reciprocal, the result is always 1:

$$x \times \frac{1}{x} = 1$$

2. Next, use division to find the reciprocal of a number.

   Example 1: The reciprocal of $5 = \frac{1}{5}$

   or $1 \div 5$:

$$\begin{array}{r|l} & 0.2 \\ \hline 5 & 10 \\ - & 10 \\ \hline & 0 \end{array}$$

   Which is $= 0.2$

   Example 2: The reciprocal of -3 is $\frac{1}{-3}$.

   or $1 \div -3$:

$$
\begin{array}{r|l}
 & 0.333 \\
\hline
3 & 10 \\
- & 9 \\
\hline
 & 10 \\
- & 9 \\
\hline
 & 10 \\
- & 9 \\
\hline
 & 1
\end{array}
$$

Hence the reciprocal of -3 using division is $= -0.333...$ also written as $-0.\dot{3}$ to mean 3 is recurring.

Example 1.1.19 Find the reciprocal of the following numbers using division.

1. 256
2. 4.2

Solution.

1. The reciprocal of 256 is $\frac{1}{256}$.
   or: $1 \div 256$

$$
\begin{array}{r|r}
 & 0.00390625 \\
\hline
256 & 1000 \\
 & -768 \\
\hline
 & 2320 \\
- & 2304 \\
\hline
 & 1600 \\
- & 1536 \\
\hline
 & 640 \\
- & 512 \\
\hline
 & 1280 \\
- & 1280 \\
\hline
 & 0
\end{array}
$$

   Hence, the reciprocal of 256 using division is $1 \div 256 = 0.00390625$

2. The reciprocal of 4.2 according to definition is $\frac{1}{4.2}$.
   We can rerwrite $\frac{1}{4.2}$ as $\frac{1}{4.2} \times \frac{10}{10}$

$$
\begin{aligned}
&= \frac{10}{42} \\
&= \frac{5}{21}
\end{aligned}
$$

   Dividing 5 by 21 we have:

$$
\begin{array}{rl}
 & 0.238095 \\ \cline{2-2}
21 & \big) \; 50 \\
- & 42 \\ \cline{2-2}
 & 80 \\
- & 63 \\ \cline{2-2}
 & 170 \\
- & 168 \\ \cline{2-2}
 & 200 \\
- & 189 \\ \cline{2-2}
 & 110 \\
- & 105 \\ \cline{2-2}
 & 5
\end{array}
$$

Therefore, the reciprocal of 4.2 given to 4 decimal places is 0.2381

□

Exercise

1. Using division, find the reciprocal of following numbers:

a) 5

b) -3

c) 2.5

d) -0.4

e) $\frac{2}{7}$

f) $-\frac{5}{8}$

Answer.

a) $\frac{1}{5} = 0.2$

b) $\frac{1}{-3} = -0.333...$

c) $\frac{1}{2.5} = 0.4$

d) $\frac{1}{-0.4} = -2.5$

e) $\frac{1}{\frac{2}{7}} = \frac{7}{2} = 3.5$

f) $\frac{1}{-\frac{5}{8}} = -\frac{8}{5} = -1.6$

2. If the reciprocal of x is $\frac{1}{6}$, find the value of x.

Answer. $\frac{1}{x} = \frac{1}{6}$. Therefore, $x = 6$.

3. Movin, a Grade 10 learner covers $\frac{1}{x}$ km from home to school. If the distance he covers each day from home to school is 0.2 km. Find the value of x.

Answer. $\frac{1}{x} = \frac{2}{10} = \frac{1}{5}$ which implies that $x = 5$.

4. A cyclist covers a distance of 12 km in 1 hour. Using division to find reciprocals, determine the time taken to cover 1 km at the same speed.

Answer. The reciprocal of 12 is $1 \div 12 = \frac{1}{12}$ hours.

5. A factory machine produces $\frac{5}{8}$ of a widget every minute. How long will it take the machine to produce 40 widgets?

Answer. The reciprocal of $\frac{5}{8}$ is $1 \div \frac{5}{8} = \frac{8}{5}$ minutes per widget. Time taken to produce 40 widgets is 64 minutes.

6. A typist can type $\frac{4}{7}$ of a page in 8 minutes. At this rate, how long will it take to type a full 25-page document?

Answer. The reciprocal of $\frac{4}{7}$ is $1 \div \frac{4}{7} = \frac{7}{4}$ minutes per page. Time taken to type 25 pages is 43.75 minutes.

1.1.4 Finding reciprocals of real numbers using mathematical tables and calculators

Mathematical tables and calculators provide alternative methods for quickly determining reciprocals, especially when dealing with large numbers or decimal values. In this section, we will explore how to use both tools effectively.

1.1.4.1 Finding reciprocals of numbers using mathematical tables

Mathematical tables has reciprocal tables which are specially designed tables that allow you to quickly find the reciprocal of a number without doing any division. Each table lists a number along with its reciprocal, so you can look up the reciprocal directly.

Activity 1.1.5

Each group should have a mathematical table

1. Working in groups of 5, consider the following numbers:

a. $\frac{3}{4}$

b. $\frac{1}{3}$

c. 6

d. 0.4167

2. Discuss how to use reciprocal tables to find reciprocals of each of the given numbers.
3. Determine the reciprocals of the numbers.
4. What did you realize?
5. Share your work with fellow learners.

Key Takeaway 1.1.20 ***How to find reciprocals of numbers from the table.***

1. Given a large number e.g 1252, express the number in standard form: 1.252×10^3.
2. Next, you find the reciprocal of the number from the reciprocal table as follows.

9.0 RECIPROCALS

											SUBTRACT								
x	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
1.0	1.0000	9901	9804	9709	9615	9524	9434	9346	9259	9174	9	18	28	37	46	55	64	73	83
1.1	0.9091	9009	8929	8850	8772	8696	8621	8547	8475	8403	8	15	23	31	38	46	53	61	69
1.2	0.8333	8264	8197	8130	8065	8000	7937	7874	7813	7752	6	13	19	26	32	39	45	52	58
1.3	0.7692	7634	7576	7519	7463	7407	7353	7299	7246	7194	6	11	17	22	28	33	39	44	50
1.4	0.7143	7092	7042	6993	6944	6897	6849	6803	6757	6711	5	10	14	19	24	29	34	38	43
1.5	0.6667	6623	6579	6536	6494	6452	6410	6369	6329	6289	4	8	13	17	21	25	29	34	38
1.6	0.6250	6211	6173	6135	6098	6061	6024	5988	5952	5917	4	7	11	15	18	22	26	30	33
1.7	0.5882	5848	5814	5780	5747	5714	5682	5650	5618	5587	3	7	10	13	16	20	23	26	30
1.8	0.5556	5525	5495	5464	5435	5405	5376	5348	5319	5291	3	6	9	12	15	17	20	23	26
1.9	0.5263	5236	5208	5181	5155	5128	5102	5076	5051	5025	3	5	8	10	13	16	18	21	23
2.0	0.5000	4975	4950	4926	4902	4878	4854	4831	4808	4785	2	5	7	10	12	14	17	19	22
2.1	0.4762	4739	4717	4695	4673	4651	4630	4608	4587	4566	2	4	7	9	11	13	15	18	20
2.2	0.4545	4525	4505	4484	4464	4444	4425	4405	4386	4367	2	4	6	8	10	12	14	16	18
2.3	0.4348	4329	4310	4292	4274	4255	4237	4219	4202	4184	2	4	5	7	9	11	13	14	16
2.4	0.4167	4149	4132	4115	4098	4082	4065	4049	4032	4016	2	3	5	7	8	10	12	14	15
2.5	0.4000	3984	3968	3953	3937	3922	3906	3891	3876	3861	2	3	5	6	8	9	11	12	14
2.6	0.3846	3831	3817	3802	3788	3774	3759	3745	3731	3717	1	3	4	6	7	8	10	11	13
2.7	0.3704	3690	3676	3663	3650	3636	3623	3610	3597	3584	1	3	4	5	7	8	9	10	12
2.8	0.3571	3559	3546	3534	3521	3509	3497	3484	3472	3460	1	2	4	5	6	7	8	10	11
2.9	0.3448	3436	3425	3413	3401	3390	3378	3367	3356	3344	1	2	4	5	6	7	8	10	11
3.0	0.3333	3322	3311	3300	3289	3279	3268	3257	3247	3236	1	2	3	4	5	7	8	9	10
3.1	0.3226	3215	3205	3195	3185	3175	3165	3155	3145	3135	1	2	3	4	5	6	7	8	9
3.2	0.3125	3115	3106	3096	3086	3077	3067	3058	3049	3040	1	2	3	4	5	6	7	8	9
3.3	0.3030	3021	3012	3003	2994	2985	2976	2967	2959	2950	1	2	3	4	4	5	6	7	8

a) Move down the column headed x to locate 1.2

b) Move to the right along the row that has 1.2 to the column headed 5.

c) Read the number at the intersection of the row and column which is 0.8000

d) Move farther to the right on the same row to the SUBTRACT column headed 2.

e) Read the number at the intersection of the row and column which is 13.

f) Subtract the number 13 from 0.800. While subtracting align 13 to the right as shown below.

$$\begin{array}{r} 0\ .\ 8\ 0\ 0 \\ -\ 0\ .\ 0\ 1\ 3 \\ \hline 0\ .\ 7\ 8\ 7 \\ \hline \end{array}$$

4. Calculate the reciprocal of 10^3.
Since 10^3 can be written as 1000 then its reciprocal is $\frac{1}{1000}$.
5. Multiply 0.787 by $\frac{1}{1000}$.
6. Hence, the reciprocal of 1252 is given by $0.787 \times \frac{1}{1000} = 0.000787$

Example 1.1.21 Use tables to find the reciprocal of 0.154.

Solution. Given 0.154, first write it in standard form: 1.54×10^{-1}.

9.0 RECIPROCALS

											SUBTRACT								
x	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
1.0	1.0000	9901	9804	9709	9615	9524	9434	9346	9259	9174	9	18	28	37	46	55	64	73	83
1.1	0.9091	9009	8929	8850	8772	8696	8621	8547	8475	8403	8	15	23	31	38	46	53	61	69
1.2	0.8333	8264	8197	8130	8065	8000	7937	7874	7813	7752	6	13	19	26	32	39	45	52	58
1.3	0.7692	7634	7576	7519	7463	7407	7353	7299	7246	7194	6	11	17	22	28	33	39	44	50
1.4	0.7143	7092	7042	6993	6944	6897	6849	6803	6757	6711	5	10	14	19	24	29	34	38	43
1.5	0.6667	6623	6579	6536	6494	6452	6410	6369	6329	6289	4	8	13	17	21	25	29	34	38
1.6	0.6250	6211	6173	6135	6098	6061	6024	5988	5952	5917	4	7	11	15	18	22	26	30	33
1.7	0.5882	5848	5814	5780	5747	5714	5682	5650	5618	5587	3	7	10	13	16	20	23	26	30
1.8	0.5556	5525	5495	5464	5435	5405	5376	5348	5319	5291	3	6	9	12	15	17	20	23	26
1.9	0.5263	5236	5208	5181	5155	5128	5102	5076	5051	5025	3	5	8	10	13	16	18	21	23
2.0	0.5000	4975	4950	4926	4902	4878	4854	4831	4808	4785	2	5	7	10	12	14	17	19	22
2.1	0.4762	4739	4717	4695	4673	4651	4630	4608	4587	4566	2	4	7	9	11	13	15	18	20
2.2	0.4545	4525	4505	4484	4464	4444	4425	4405	4386	4367	2	4	6	8	10	12	14	16	18
2.3	0.4348	4329	4310	4292	4274	4255	4237	4219	4202	4184	2	4	5	7	9	11	13	14	16
2.4	0.4167	4149	4132	4115	4098	4082	4065	4049	4032	4016	2	3	5	7	8	10	12	14	15
2.5	0.4000	3984	3968	3953	3937	3922	3906	3891	3876	3861	2	3	5	6	8	9	11	12	14
2.6	0.3846	3831	3817	3802	3788	3774	3759	3745	3731	3717	1	3	4	6	7	8	10	11	13
2.7	0.3704	3690	3676	3663	3650	3636	3623	3610	3597	3584	1	3	4	5	7	8	9	10	12
2.8	0.3571	3559	3546	3534	3521	3509	3497	3484	3472	3460	1	2	4	5	6	7	8	10	11
2.9	0.3448	3436	3425	3413	3401	3390	3378	3367	3356	3344	1	2	4	5	6	7	8	10	11
3.0	0.3333	3322	3311	3300	3289	3279	3268	3257	3247	3236	1	2	3	4	5	7	8	9	10
3.1	0.3226	3215	3205	3195	3185	3175	3165	3155	3145	3135	1	2	3	4	5	6	7	8	9
3.2	0.3125	3115	3106	3096	3086	3077	3067	3058	3049	3040	1	2	3	4	5	6	7	8	9
3.3	0.3030	3021	3012	3003	2994	2985	2976	2967	2959	2950	1	2	3	4	4	5	6	7	8

Then, to locate the reciprocal of the given number:

i) Move down the column headed x to locate 1.5.

ii) Move to the right along the row that has 1.5 to the column headed 4.

iii) Read the number at the intersection of the row and column which is 0.6494.

Calculate the reciprocal of 10^{-1}.
Since 10^{-1} can be written as $\frac{1}{10}$ then its reciprocal is $\frac{10}{1} = 10$.
Multiply 0.6494 by 10.
The reciprocal of 0.154 is 6.494 □

Example 1.1.22 Murunga's car consumes $\frac{1}{8}$ liters of fuel per kilometer. Use tables to identify how far Murunga can drive with 1 liter?

Solution. If $\frac{1}{8}$liters $=$ 1km,
Then 1liter =? km
To find the distance Murunga drove using 1 liter, we first convert $\frac{1}{8}$ to a decimal i.e $1 \div 8 = 0.125$.
Next, we write 0.125 in standard form: 1.25×10^{-1}.

9.0 RECIPROCALS

											SUBTRACT								
x	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
1.0	1.0000	9901	9804	9709	9615	9524	9434	9346	9259	9174	9	18	28	37	46	55	64	73	83
1.1	0.9091	9009	8929	8850	8772	8696	8621	8547	8475	8403	8	15	23	31	38	46	53	61	69
1.2	0.8333	8264	8197	8130	8065	8000	7937	7874	7813	7752	6	13	19	26	32	39	45	52	58
1.3	0.7692	7634	7576	7519	7463	7407	7353	7299	7246	7194	6	11	17	22	28	33	39	44	50
1.4	0.7143	7092	7042	6993	6944	6897	6849	6803	6757	6711	5	10	14	19	24	29	34	38	43
1.5	0.6667	6623	6579	6536	6494	6452	6410	6369	6329	6289	4	8	13	17	21	25	29	34	38
1.6	0.6250	6211	6173	6135	6098	6061	6024	5988	5952	5917	4	7	11	15	18	22	26	30	33
1.7	0.5882	5848	5814	5780	5747	5714	5682	5650	5618	5587	3	7	10	13	16	20	23	26	30
1.8	0.5556	5525	5495	5464	5435	5405	5376	5348	5319	5291	3	6	9	12	15	17	20	23	26
1.9	0.5263	5236	5208	5181	5155	5128	5102	5076	5051	5025	3	5	8	10	13	16	18	21	23
2.0	0.5000	4975	4950	4926	4902	4878	4854	4831	4808	4785	2	5	7	10	12	14	17	19	22
2.1	0.4762	4739	4717	4695	4673	4651	4630	4608	4587	4566	2	4	7	9	11	13	15	18	20
2.2	0.4545	4525	4505	4484	4464	4444	4425	4405	4386	4367	2	4	6	8	10	12	14	16	18
2.3	0.4348	4329	4310	4292	4274	4255	4237	4219	4202	4184	2	4	5	7	9	11	13	14	16
2.4	0.4167	4149	4132	4115	4098	4082	4065	4049	4032	4016	2	3	5	7	8	10	12	14	15
2.5	0.4000	3984	3968	3953	3937	3922	3906	3891	3876	3861	2	3	5	6	8	9	11	12	14
2.6	0.3846	3831	3817	3802	3788	3774	3759	3745	3731	3717	1	3	4	6	7	8	10	11	13
2.7	0.3704	3690	3676	3663	3650	3636	3623	3610	3597	3584	1	3	4	5	7	8	9	10	12
2.8	0.3571	3559	3546	3534	3521	3509	3497	3484	3472	3460	1	2	4	5	6	7	8	10	11
2.9	0.3448	3436	3425	3413	3401	3390	3378	3367	3356	3344	1	2	4	5	6	7	8	10	11
3.0	0.3333	3322	3311	3300	3289	3279	3268	3257	3247	3236	1	2	3	4	5	7	8	9	10
3.1	0.3226	3215	3205	3195	3185	3175	3165	3155	3145	3135	1	2	3	4	5	6	7	8	9
3.2	0.3125	3115	3106	3096	3086	3077	3067	3058	3049	3040	1	2	3	4	5	6	7	8	9
3.3	0.3030	3021	3012	3003	2994	2985	2976	2967	2959	2950	1	2	3	4	4	5	6	7	8

Move down the column headed x to locate 1.2.
Move to the right along the row that has 1.2 to the column headed 5.
Read the number at the intersection of the row and column which is 0.8000.
Calculate the reciprocal of 10^{-1}.
Since 10^{-1} can be written as $\frac{1}{10}$ then its reciprocal is $\frac{10}{1} = 10$.
Multiply 0.8000 by 10.
$0.8000 \times 10 = 8$
Therefore, Murunga can drive 8 km using 1 l of fuel. □

Exercise

1. Find the reciprocals of the following numbers using reciprocal tables:

a) 4286
b) 0.0458
c) 0.007582
d) 2.781
e) $\frac{3}{8}$
f) $\frac{4}{0.125}$

Answer 1.

a) 0.0002332

b) 21.83

c) 131.9

d) 0.3596

e) 26.67

f) 0.03125

Answer 2.

2. Use mathematical tables to evaluate each of the following:

a) $\frac{100}{29.56}$
b) $\frac{1}{\sqrt{0.2704}}$
c) $\frac{1}{1.374^2}$
d) $1000 \times \frac{1}{0.7598}$
e) $\frac{3}{\sqrt{2025}}$
f) $3.054 + \frac{1}{60.84}$

Answer.

a) $\frac{100}{29.56} = 100 \times \frac{1}{29.56}$
$= 2.956 \times 10^1 = 0.3383 \times 10^{-1} = 0.03383$
$100 \times 0.03383 = 3.383$

b) $\frac{1}{\sqrt{0.2704}} = \frac{1}{0.52}$
$5.2 \times 10^{-1} = 0.1923 \times 10^1 = 1.923$

c) $\frac{1}{1.374^2} = \frac{1}{1.887876}$
$1.888 = 0.5296$

d) $1000 \times \frac{1}{0.7598} = 1000 \times \frac{1}{7.598 \times 10^{-1}}$
$7.598 = 0.1317 \times 10^1 = 1.317$
$1000 \times 1.317 = 1317$

e) $\frac{5}{\sqrt{2025}} = \frac{5}{45} = \frac{1}{9}$
reciprocal of 9 is 0.1111

f) $3.054 + \frac{1}{60.84} = 3.054 + \frac{1}{6.084 \times 10^1}$
$6.084 = 0.1644 \times 10^{-1} = 0.01644$
$3.054 + 0.01644 = 3.07044$

3. Maria conducted a survey to find out which games students enjoy. She discovered that 0.6 of the students preferred playing football. Use mathematical tables to find the reciprocal of 0.6.

Answer. $0.6 = 6 \times 10^{-1}$
Reciprocal of 0.6 is $0.1667 \times 10^{1} = 1.667$

4. A farmer can plant 0.85 acres of land in one day. How many days will it take the farmer to plant one acre?(Hint: Use the reciprocal of the planting rate.)Use mathematical tables to find your answer.

Answer. Number of days by acre is given by $\frac{1}{0.85}$
$0.85 = 8.5 \times 10^{-1}$
$0.1176 \times 10^{1} = 1.176$

5. John can type 80 words per minute.

a) How many minutes does it take him to type one word?

b) Based on your answer, how long will it take him to type a 400-word essay?

Answer.

6. A shopkeeper has 96 apples and packs them in groups of 8 apples per pack.

a) How many packs does the shopkeeper make?

b) What is the reciprocal of the number of apples per pack?

Answer.

1.1.4.2 Finding reciprocals using a calculator

Activity 1.1.6

1. Working in groups, use a calculator to work out the reciprocal of 151.6

i) Press the keys [1], [÷], [1] ,[5] ,[1], [.] ,[6] in that order.

ii) Press the key [=]

2. Read the displayed result. What is the reciprocal of 151.6 from the calculator?

3. Work out the reciprocal of each of the following numbers using the calculator:

a) 0.0038

b) 0.5498

c) $\frac{1}{8}$

d) 564

4. Discuss with other learners how you determine the reciprocal of a number using a calculator.

Key Takeaway 1.1.23 To find reciprocal of a number using a calculator:

1. Enter the number into the calculator,
2. Press the reciprocal button x^{-1} or divide 1 by the number.
3. Read the displayed result.

Example: To find the reciprocal of 7,

i) Press the key 1
ii) Press the key ÷
iii) Press the key 7
iv) Press the key =

Read the displayed result which is 0.14285714286. Hence the reciprocal of 7 given to 4 decimal places is 0.1429.

Example 1.1.24 Find the reciprocal of the following using a calculator.

i) 5.6 ii) 0.003 iii) 12.8

Solution. i) To find the reciprocal of 5.6, press the keys 1, ÷, 5, . then 6.

Press the key =

Read the reciprocal of 5.6 on the screen of the calculator.

Hence the reciprocal of 5.6 from the calculator is approximately 0.1786, when rounded off to 4 decimal places.

ii) To calculate the reciprocal of 0.003 :

Press the keys 1, ÷, 0, ., 0, 0, then 3.

Press the key [=]
Read the reciprocal of 0.003 on the screen of the calculator.
Therefore, the reciprocal of 0.003 from the calculator is: 333.333333.... Also written as 333.3333 when rounded off to 4 decimal places.
iii) To calculate the reciprocal of 12.8:
Press the keys [1], [÷], [1], [2], [.] then [8].
Press the key [=]
Read the reciprocal of 12.8 on the screen of the calculator.
Hence the reciprocal of 12.8 from the calculator is: 0.078125. □

Checkpoint 1.1.25 Using a Calculator to Find Reciprocals. Load the question by clicking the button below.Use a calculator to find the reciprocal of each number. Leave your answers to ***4 decimal places***.

1. 191 ______
2. 0.4433 ______

Answer 1. 0.0052

Answer 2. 2.2558

Solution. Worked Solution To find reciprocal of 191 and 0.4433 using a calculator:

1. Enter the number into the calculator
2. Press the reciprocal button x□¹ on the calculator
3. Read the displayed result.

1. Therefore, the reciprocal of 191 is (0.00523560209424 = 0.0052 when rounded off.
2. The reciprocal of 0.4433 is 2.25580870742 . = 2.2558

Checkpoint 1.1.26 Finding Reciprocals of Rational and Irrational Numbers. Load the question by clicking the button below.Find the reciprocal of the following numbers using a calculator. Leave your answer to 4 decimal places where possible.
a) $\sqrt{440} =$ ______
b) $\frac{2}{\pi} =$ ______
c) $\frac{7}{26} =$ ______

Answer 1. 0.0477

Answer 2. 1.5708

Answer 3. $\frac{26}{7}$

Solution. Worked Solution To find reciprocal of $\sqrt{440}$, $\frac{2}{\pi}$ and $\frac{7}{26}$ using a calculator:

1. Enter the number into the calculator
2. Press the reciprocal button x□¹ on the calculator
3. Press = sign and read the displayed result.

a) Therefore, the reciprocal of $\sqrt{440}$ is 0.0476731294623
Given to 4 decimal places we have 0.0477
b) The reciprocal of $\frac{2}{\pi}$ is 1.57079632679 Giving the number to 4 decimal points we have: 1.5708
c) Given $\frac{7}{26}$, the reciprocal is $\frac{26}{7}$

Exercise

1. Find the reciprocal of the following numbers using a calculator:

a) 8
b) 125
c) 598
d) 8638
e) 8.861
f) 0.00067
g) 0.01467
h) 0.4875

Answer. The reciprocals are:

a) 0.125
b) 0.008
c) 0.001672240802675585
d) 0.000115767538782125
e) 0.11285407967498025
f) 1492.537313432835821
g) 68.16632583503749148
h) 2.051282051282051282

2. A school cafeteria has 8 large trays of food to serve equally among the students. To find out how much food each student will get per tray, the cafeteria manager needs to calculate the reciprocal of 8. Using a calculator, how can the manager find the reciprocal of 8, and what is the answer?

Answer. The manager can find the reciprocal of 8 by pressing the keys [1], [÷], [8], then [=]. The answer displayed on the calculator is 0.125.

3. A group of friends has 5 bottles of juice to share equally. Use a calculator to determine how much juice each person gets per bottle.

Answer. Calculate the reciprocal of 5 which is 0.2.

4. If a machine completes a task in 6 hours, its work rate per hour is the reciprocal of the time. Use a calculator to determine the reciprocal and explain what it represents.

Answer. The reciprocal of 6 is 0.1667 (rounded to four decimal places). This represents the fraction of the task the machine completes in one hour.

5. A car travels 12 kilometers on 1 liter of fuel. Use a calculator to find out liters of fuel needed per kilometer.

Answer. The reciprocal of 12 is $\frac{1}{12}$ which is 0.0833 (rounded to four decimal places). This means the car uses approximately 0.0833 liters of fuel per kilometer.

6. Write down 3 numbers and work out their reciprocals using a calculator.

Answer. The reciprocals of the numbers is calculated using a calculator by entering 1 divided by each number you wrote down.

1.1.5 Using reciprocals of real numbers in mathematical computations

This is application of the concept of reciprocals to perform different types of mathematical operations such as division, solving equations, proportions, rates, and real-life problem-solving.

Activity 1.1.7

In groups, find an exercise book and a pen.

1. Write different numbers (that is positive, negative, decimal numbers and fractions) on the exercise book.

Example: $2, -4, 0.25, \frac{5}{8}$

2. Find the reciprocals of the numbers you have formed.

Example: Number formed is $\frac{5}{8}$ its reciprocal is given by $1 \div \frac{5}{8} = 1 \times \frac{8}{5} = \frac{8}{5}$

3. Multiply the original number you formed by its corresponding reciprocal and observe the result.

Example: $\frac{5}{8} \times \frac{8}{5}$

4. Discuss cases where reciprocals do not exist.

5. Identify a real world problem where reciprocals are useful.

6. Share your work with fellow learners.

Key Takeaway 1.1.27 ***How reciprocals are used in mathematical computations:***

1. Converting Division into Multiplication

   Instead of dividing a number, we multiply by the reciprocal: $a \div b = a \times \frac{1}{b}$.

   Example with whole numbers:

   $8 \div 2 = 8 \times \frac{1}{2} = 4$

   Example with fractions:

$$\begin{aligned}\frac{5}{6} \div \frac{2}{3} =& \frac{5}{6} \times \frac{3}{2} \\ =& \frac{15}{12} \\ =& \frac{5}{4}\end{aligned}$$

2. Solving Equations

   When solving equations where a variable is multiplied by a number, we use the reciprocal to isolate the variable.

   Example: Solve for x in the equation $3x = 12$

   To solve for x, multiply both sides by the reciprocal of 3.

   Therefore, $x = 12 \times \frac{1}{3} = 4$

3. Working with proportions and ratios while cooking.

   Example: If a recipe uses $\frac{2}{3}$ of a cup of sugar per serving, how many servings can you make with 4 cups?

   To find the number of servings, compute: $4 \div \frac{2}{3}$

$$\begin{aligned}=& 4 \times \frac{3}{2} \\ =& \frac{12}{2} \\ =& 6 \text{ servings}\end{aligned}$$

4. Solving real-world problems.

   1. A printing machine can print $\frac{5}{6}$ of a book page per second. How long will it take to print 20 pages?

If 1 second = $\frac{5}{6}$ of a page, then for 20 pages, we calculate as follows:

$$\begin{aligned}\text{Time } =&20 \div \frac{5}{6}\\ =&20 \times \frac{6}{5}\end{aligned}$$

Multiplying and simplifying we have: Time = $\frac{120}{5}$ = 24 seconds.

Thus, the printing machines will take 24 seconds to print 20 pages.

2. If a car travels at 80 km/h, to find out the time taken per km we can use reciprocal.

Time per km = $\frac{1}{80}$ hours per km.

3. If a factory produces 300 items in 5 hours, then the production rate per hour is:

$\frac{300}{5}$ = 60 items per hour.

To determine how much time is needed per item, we use reciprocal:

Hence, the factory produces 1 item every $\frac{1}{60}$ hours.

Example 1.1.28 Kerich is a farmer and has $\frac{2}{3}$ of an acre of land and wants to divide it into plots of $\frac{1}{6}$ acre each. How many plots can Kerich make?

Solution. To find the number of plots that Kerich can make, we solve it as a division problem.

Divide $\frac{2}{3}$ by $\frac{1}{6}$ and solve it by multiplying $\frac{2}{3}$ by the reciprocal of $\frac{1}{6}$ as below:

$$\begin{aligned}\frac{2}{3} \div \frac{1}{6} =&\frac{2}{3} \times \frac{6}{1}\\ \frac{12}{3} =&4\end{aligned}$$

Hence Kerich can make 4 plots. □

Example 1.1.29 Njoki, a business woman invest ksh 5 000 in a business that promises to double his investment every 1.5 years. What fraction of her initial investment will she have after 4.5 years?

Solution. If 1.5 years = 2 times the initial investment, how about 4.5 years?

Computing the number of times her investment will yield after 4.5 years we have: $4.5 \div 1.5$

$$\begin{aligned}&=4.5 \times \frac{1}{1.5}\\ &=3\end{aligned}$$

This means her investment will double 3 times.
Therefore, after 4.5 years her investment will be $5000 \times 2 \times 2 \times 2$

$$\begin{aligned}&=5000 \times 8\\ &=40\,000\end{aligned}$$

Illustration

Representing the number of years with their corresponding double investment in ksh makes it easier to understand the above computations.

Remember at the start of a period we will represent it with a 0.

Years:	0	$\rightarrow$	1.5	$\rightarrow$	3	$\rightarrow$	4.5
Shillings:	5 000	$\rightarrow$	10 000	$\rightarrow$	20 000	$\rightarrow$	40 000

Hence Njoki will have ksh 40 000 after 4.5 years. ☐

Example 1.1.30 Okoth runs $3\frac{1}{4}$ miles in $\frac{1}{2}$ an hour. What is her speed in miles per hour?

Solution. To find speed we divide distance by time such that: speed $= \frac{\text{distance}}{\text{time}}$.

First, we need to convert the fraction $3\frac{1}{4}$ into an improper fraction which is $\frac{13}{4}$

Hence, speed $= 3\frac{1}{4} \div \frac{1}{2} = \frac{13}{4} \div \frac{1}{2}$

Multiply $\frac{13}{4}$ by the reciprocal of $\frac{1}{2}$.

$\frac{13}{4} \times \frac{2}{1} = \frac{26}{4}$

Simplifying $\frac{26}{4}$ we get $\frac{13}{2}$ which is $= 6\frac{1}{2}$. ☐

Exercise

1. Solve the following using reciprocals:

a) $6 \div \dfrac{2}{3}$

b) $4x = 12$

c) $\dfrac{3}{5} \times x = 9$

d) $0.2 \div 4.5$

Answer.

a) $6 \div \dfrac{2}{3} = 6 \times \dfrac{3}{2} = 9$

b) $x = 12 \times \dfrac{1}{4} = 3$

c) $x = 9 \times \dfrac{5}{3} = 15$

d) $0.2 \div 4.5 = 0.2 \times \dfrac{1}{4.5} = \dfrac{2}{45} = 0.0444\cdots$

2. Solve: $\left(\frac{4}{7} \div \frac{2}{5}\right) \times \left(\frac{3}{8} \div \frac{9}{16}\right)$

Answer.

$$\begin{aligned}\left(\frac{4}{7} \div \frac{2}{5}\right) \times \left(\frac{3}{8} \div \frac{9}{16}\right) &= \left(\frac{4}{7} \times \frac{5}{2}\right) \times \left(\frac{3}{8} \times \frac{16}{9}\right) \\ &= \frac{20}{14} \times \frac{48}{72} \\ &= \frac{10}{7} \times \frac{2}{3} \\ &= \frac{20}{21}\end{aligned}$$

3. If $x \times \frac{3}{8} = \frac{5}{12}$, find the value of x using reciprocals.

Answer. $x = \frac{5}{12} \times \frac{8}{3} = \frac{40}{36} = \frac{10}{9}$

4. A rope of length 18 m is cut into pieces each measuring $\frac{3}{4}$ m. How many pieces are obtained?

Answer. $18 \div \frac{3}{4} = 18 \times \frac{4}{3} = 24$ pieces.

5. A hiker is walking 24 km and takes breaks every $\frac{2}{3}$ km. How many breaks will he take by the end of the journey? If he changes his break interval to $1\frac{1}{3}$ km, how does the number of breaks change?

Answer. Number of breaks at $\frac{2}{3}$ km intervals: $24 \div \frac{2}{3} = 24 \times \frac{3}{2} = 36$ breaks.

Number of breaks at $1\frac{1}{3}$ km intervals: $24 \div 1\frac{1}{3} = 24 \div \frac{4}{3} = 24 \times \frac{3}{4} = 18$ breaks.

Changing the break interval to $1\frac{1}{3}$ km reduces the number of breaks from 36 to 18.

6. A road is 12.6 km long and is divided into equal segments of 0.35 km for maintenance. How many segments are there?

Answer. $12.6 \div 0.35 = 12.6 \times \frac{1}{0.35} = 36$ segments.

1.2 Indices and Logarithms

1.2.1 Indices

Introduction. Mathematics involves working with very large and very small numbers, which can be simplified using indices. An *index* (plural, *indices*) also called *exponent* or *power* is a way of writing repeated multiplication of the same number.

Why Are Indices Important?

Indices are widely used in:

- Scientific notation writing, where numbers are sometimes very large and sometimes very small, such as the speed of light or the size of an atom.
- Physics and Engineering to understand exponential growth and decay rates, such as radioactive decay and population growth.
- Finance and money in calculating compound interest.

1.2.1.1 Expressing Numbers in Index Form

Numbers can be written in different ways. One useful way is index notation also called exponential notation. Index notation, is a way of writing repeated multiplication of the same number.

Exponent is commonly known as the **power**.

Activity 1.2.1 Work in groups:

Materials Needed: One $A4$ sheet of paper per student. A piece of paper is appoimately. 0.1 mm thick

Instruction:

Form groups of 2 or −3 students.

- Take a piece of paper and fold it in half once. Count the number of layers.
- Fold it in half again and count the number of layers.
- Continue folding and record your observations in a table.
- Express the number of layers using index notation.

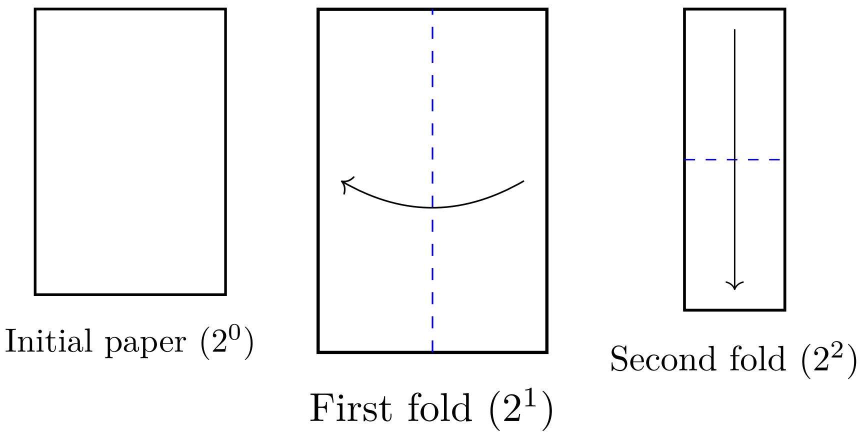


Figure 1.2.1 Illustrating the folding of paper to represent indices

Table 1.2.2 Table Showing the Growth of Paper Thickness with Each Fold

Number of Folds	Thickness (mm)	Index Notation
0 (no fold)	0.1 mm	$2^0 \times 0.1$
1 fold	0.2 mm	$2^1 \times 0.1$
2 folds	0.4 mm	$2^2 \times 0.1$
3 folds	?	?
4 folds	?	?
5 folds	?	?

a) What pattern do you observe in the numbers of layers?

b) Can you express the pattern using indices?

c) If you could fold the paper 10 times, how thick would it be?

d) Can you think of other situations where things double repeatedly?

When writing Index Notation we represent it as:

$$a^n$$

Where:
a is the base (the number being multiplied).
n is the exponent/ index /power (the number of times the base is multiplied by itself).

Example:
$2a = a \times a = a^2$
$8 = 2 \times 2 \times 2$
$= 2^3$
$625 = 5 \times 5 \times 5 \times 5$
$= 5^4$

Checkpoint 1.2.3 Expressing Whole Numbers in Index Form. Load the question by clicking the button below.Write the number 729 in index form.

Note:Your answer should be written as a^b.

______

Answer. 729

Solution. Worked Solution The number given is 729.

We check if it can be written as repeated multiplication of the same base.

$729 = 9 \times 9 \times \cdots \times 9 \ \ (3 \ \ times)$
Therefore, the index form is

$$9^3.$$

Checkpoint 1.2.4 Expressing Cube Power Results in Index Form. Load the question by clicking the button below.A school has a computer storage server that saves student work in blocks. Each block holds repeated copies of the same unit file. One unit file has a size of 3 MB. After processing, the total saved size is 243 MB.

Write 243 in index form using base 3.

____

Answer. 243

Solution. Worked Solution The total saved size is 243 MB.
We check how many times 3 appears as a repeated factor:
$243 = 3 \times 3 \times \cdots \times 3$ (repeated 5 times)
Thus, the index form is:

$$3^5.$$

1.2.1.2 Laws of Indices

When we need to work with indices efficiently, we use a set of rules called the Laws of Indices. They are useful in carrying out operations involving indices.

a) Product Law
When multiplying numbers with the same base, add their powers.
i.e $a^m \times a^n = a^{m+n}$
Example:

$$3^2 \times 3^4 = 3^{2+4}$$

$$= 3^6$$

b) Quotient Law
When dividing numbers with the same base, subtract their powers.
i.e $\frac{a^m}{a^n} = a^{m-n}$
Example:

$$5^7 \div 5^3 = \frac{5^7}{5^3} = \frac{5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5}{5 \times 5 \times 5}$$

$$= 5^{(7-3)}$$

$$= 5^4$$

c) Power of a Power Law
When raising a power to another power, multiply the powers.
i.e $(a^m)^n = a^{(m \times n)}$
Example:

$$\left(3^2\right)^3 = (3 \times 3) \times (3 \times 3) \times (3 \times 3)$$

$$= 3^{(2\times 3)}$$

$$= 3^6$$

d) Power of a Product Law

When raising a power to a product factor, multiply the power to all factors inside the bracket.

i.e $(ab)^n = a^n \times b^n$

Example:

$$\begin{aligned}(2 \times 3)^3 =& 2^3 \times 3^3 \\ =& 8 \times 27 \\ =& 216\end{aligned}$$

e) Fractional Powers

A fractional power represents a root.

i.e $(a^{\frac{m}{n}})$

$$(a^m)^{\frac{1}{n}} = \sqrt[n]{a^m}$$

Example:

Solve:

i) $8^{\frac{1}{3}}$

$$= \sqrt[3]{8^1}$$

$$= 2$$

Similarly we can say,

$$8 = 2 \times 2 \times 2 = 2^3$$

$$8^{\frac{1}{3}} = (2^3)^{\frac{1}{3}}$$

$$= 2^{(3 \times \frac{1}{3})}$$

$$= 2$$

ii) $(\frac{4}{5})^3$

$$(\frac{a}{b})^3 = \frac{a^3}{b^3}$$

$$(\frac{4}{5})^3 = \frac{4^3}{5^3}$$

$$= \frac{64}{125}$$

Table 1.2.5 Summary of the Laws of Indices

Law	*Expression*	*Example*
Product Law	$a^m \times a^n = a^{(m+n)}$	$2^3 \times 2^4 = 2^{(3+4)} = 2^7 = 128$
Quotient Law	$a^m \div a^n = a^{(m-n)}$	$5^7 \div 5^3 = 5^{(7-3)} = 5^4$
Power of a Power Law	$(a^m)^n = a^{(m\times n)}$	$(3^2)^3 = 3^{(2\times 3)} = 3^6$
Power of a Product Law	$(a \times b)^n = a^n \times b^n$	$(2 \times 3)^3 = 2^3 \times 3^3 = 216$
Fractional Powers	$a^{(\frac{m}{n})} = n\sqrt{(a^m)}$	$8^{(\frac{1}{3})} = \sqrt[3]{8} = 2$

Important

- The laws apply only when the bases are the same

Activity 1.2.2 Work in groups:

Materials Needed: A pen and a notebook

Instructions:

- Form groups of three
- Each group gets a set of index question (on flashcards or written on the board).
- Write the question on your notebook
- Simplify using the laws of indices that applies to your question.
- Each group to present their working to the class.

$6^4 \times 6^3$	$\frac{9^7}{9^5}$	$(2^3)^5$	$(4 \times 5)^2$	$\left(\frac{3}{7}\right)^4$

Example 1.2.6 Simplify the expression:

$2^3 \times 2^4$

Solution. Expand each power:

$$2^3 = (2 \times 2 \times 2)$$
$$2^4 = (2 \times 2 \times 2 \times 2)$$
$$(2 \times 2 \times 2) \times (2 \times 2 \times 2 \times 2)$$

Count the total number of factors

$$2^{(3+4)} = 2^7$$
$$= 128$$

□

Checkpoint 1.2.7 Applying the Product Law of Indices. Load the question by clicking the button below.The Agriculture Club is doing mango processing and is making mango juice. Each jar of mango juice needs 5^1 mangoes. The group makes 5^1 jars of mango juice.

How many mangoes are needed in total? Write your answer as a ***single power***, in index form.

____

Answer. 25

Solution. Worked Solution The total number of mangoes is found by multiplying the number of mangoes per jar by the number of jars.

We use the product law of indices:

$$a^m \times a^n = a^{m+n}$$

So,

$$5^1 \times 5^1 = 5^2$$

Therefore, the total number of mangoes needed is:

$$\boxed{5^2}$$

Checkpoint 1.2.8 Simplify Indices. Load the question by clicking the button below.Use the laws of indices to simplify.

1. $\dfrac{a^9 \times a^8}{a^5} =$ _______
2. $\dfrac{m^{10} \times m^{-6}}{m^{-5}} =$ _______

Answer 1. a^{12}

Answer 2. m^9

Solution.

1. Use the first law of indices to simplify the numerator:
2. Use the first law of indices to simplify the numerator:

Checkpoint 1.2.9 Applying Mixed Laws of Indices. Load the question by clicking the button below. ***Watamuu Comprehensive School*** is buying blocks of writing paper for Grade 10 learners. Each block contains 7^2 sheets of paper. The school buys 7^1 blocks. The blocks are then taken apart and its sheets are packed into bundles to be given to the students, each bundle containing 7^1 sheets.

By applying Laws of indices, how many bundles of sheets are formed? Write your answer as a ***single power***, in index form.

____

Answer. 49

Solution. Worked Solution First, find the total number of sheets:

$$7^2 \times 7^1 = 7^3$$

Next, divide by the number of sheets in each bundle:

$$\frac{7^3}{7^1} = 7^{3-1}$$

Therefore, the number of bundles formed is:

$$\boxed{7^2}$$

Checkpoint 1.2.10 Algebraic Simplification of Indices. Load the question by clicking the button below.Use the laws of indices to simplify.

1. $x^{10} \times x^{-9} =$ _______
2. $\dfrac{t^{11}}{t^{-10}} =$ _______
3. $\dfrac{y^{-5}}{y^{-8}} =$ _______

Answer 1. x

Answer 2. t^{21}

Answer 3. y^3

Solution.

1. Use the first law of indices:
2. Use the second law of indices:
3. Use the second law of indices, as for (b). $\frac{y^{-5}}{y^{-8}} = y^{-5-(-8)} = y^3$.

Exercises

1. Apply the index laws in solving the following expressions:

a) $2^6 \times 2^4$

b) $5^7 \div 5^3$

c) $(4^3)^2$

d) $\frac{9^5}{9^2} \times 9^3$

e) $8^3 \times 8^2 \div 8^4$

f) $(6^2 \times 6^3) \div 6^4$

g) $(x^3y^2)^4$

h) $\frac{m^8}{m^5} \times (m^2)^3$

2. Mr. Gitonga is a farmer. He divides his rectangular field into smaller equal plots. The total area of the field is 5^8 m^2, and each small plot has an area of 5^3 m^2. How many smaller plots does Mr. Gitonga get? Express your answer using indices.

3. Mkurugenzi Company investments grows exponentially. It was initially worth 4^5 Kenyan shillings, but after two years, it was multiplied by 4^3. Use indices to represent the total investment value after the two years.

4. Mathematics department has 2^6 books that can be borrowed and the science section has 2^4 books. Write an index expression for the total number of books in both sections if they were combined into a single shelf.

5. A carpenter is building square tables of different sizes. The first table he makes has an area of 3^2 square metres. Each new table he builds is twice the length of the previous one.

a. Express the area of the second and third tables in index form.

b. Using the Laws of Indices, find the total area of the first three tables combined.

c. If the carpenter continues doubling the table size, what will be the area of the 5^{th} table?

1.2.1.3 Zero and Negative Indices

Indices help us simplify repeated multiplication of the same number. But what happens when the exponent is zero or negative?

Zero Indices (Zero Exponent/Power Zero Law)

Activity 1.2.3 Material needed:

- Pen and paper

- A calculator

Instructions:
Look at the pattern below and complete the missing values:

$$2^4 = 16$$

$$2^3 = 8$$

$$2^2 = 4$$

$$2^1 = 2$$

$$2^0 = ?$$

From the activity;

- What pattern do you notice as the exponent decreases by 1?
- What happens when the exponent reaches zero?
- Can you try the same pattern with 3^n?

Key Takeaway:
Any non-zero number raised to power zero is always 1.
i.e $a^0 = 1$
Negative Indices

Activity 1.2.4 Look at the pattern below and complete the missing values:

$$3^3 = 27$$

$$3^2 = 9$$

$$3^1 = 3$$

$$3^0 = 1$$

$$3^{-1} = ?$$

$$3^{-2} = ?$$

- Write the answers as fractions and decimal values.
- What do you notice about the negative exponents?
- Can you try the same pattern using 5^n?

Key Takeaway:
A negative exponent means taking the reciprocal of the base.
i.e $a^{-n} = \frac{1}{a^n}$
Discussion:

- What do zero and negative indices represent, and how do they simplify expressions?
- How do negative exponents help express very small values and simplify small numbers?
- Why does any nonzero number raised to power zero equal 1? What happens when the base is zero?
- Where do we see numbers decreasing exponentially or repeated division in real life?

Example 1.2.11 The school need to make desks for grade 10 students. A carpenter is cutting wooden planks for the desks. The length of each plank decreases by half as he cuts smaller sections.

Table 1.2.12 Plank Lengths and Index Notation

Cut Number	**Plank Length (metres)**	**Index Notation**
Original Plank	1	$2^0 = 1$
First Cut	___	2^{-1}
Second Cut	___	2^{-2}
Third Cut	___	2^{-3}

Hint: The negative exponent represents how many times the length has been halved.

Solution. Each cut divides the plank by 2:

- The original plank is $2^0 = 1$ metre
- The first cut halves the plank:

$$2^{-1} = \frac{1}{2} metres$$

- The second cut halves it again:

$$2^{-2} = \frac{1}{2^2} = \frac{1}{4} metres$$

- The third cut follows the same pattern:

$$2^{-3} = \frac{1}{2^3} = \frac{1}{8} metres$$

If n represents the number of cuts, the remaining plank length L follows a trend which can bw represented as:

$$L = 2^{-n}$$

where:

$n =$ number of cuts

$L =$ remaining length after n cuts

Using this formula, let's calculate the length after 4 cuts:

$$L = 2^{-4} = \frac{1}{2^4} = \frac{1}{16} metres$$

After 4 cuts, the remaining plank length is $\frac{1}{16}$ metres

Key Takeaway

Negative indices represent repeated division.

This pattern appears in real-life situations like paper folding, battery energy life, and population decay. ☐

Checkpoint 1.2.13 Simplifying Zero Index. Load the question by clicking the button below.Simplify the expression below:

$$8b^0$$

____

Answer. 8

Solution. Worked Solution The zero index law states that any non-zero quantity raised to the power zero equals 1.

$$b^0 = 1.$$

So the expression becomes:

$$8 \times 1 = 8.$$

Checkpoint 1.2.14 Applying Negative Index Laws. Load the question by clicking the button below.Simplify the expression below using the law of negative indices:

$$6b^{-1}$$

Give your answer without negative indices.

____

Answer. $\frac{6}{b}$

Solution. Worked Solution The negative index law states:

$$a^{-n} = \frac{1}{a^n}.$$

Applying this to the expression:

$$6b^{-1} = 6 \times \frac{1}{b^1}.$$

So the simplified form is:

$$\frac{6}{b^1}.$$

Exercises

1. Fill in the missing values following the same halving pattern:

Table 1.2.15 Plank Lengths and Index Notation

Cut Number	Plank Length (meters)	Index Notation
0 (Original Plank)	1	2^0
1	1/2	2^{-1}
2	___	2^{-2}
3	___	___
4	___	___

2. In the *Msomii* Community Library, the maths-book shelf has 64 books. The librarian organizes the books by removing half of them every day.

a) Express the number of books left after 3 days using indices.

b) How many books remain after 5 days?

3. The Board of Management decided to buy a school van for Maths club in your school. The van costs Ksh $1,000,000$. Its value decreases by half every 3 years due to depreciation. Express the van's value after 9 years in index form and find how much the van will be worth after 15 years?

4. A patient takes 400 mg of a medicine. Every 4 hours, the amount of medicine in the body reduces to $\frac{1}{2}$ of what was left.

a) Write an index expression for the medicine remaining after 12 hours.

b) How much medicine remains after 20 hours?

5. A school installs 100 energy-saving bulbs. Every year, a quarter of them stop working and need replacement.

a) Write an index expression for the number of working bulbs after 4 years.

b) How many bulbs are still functional after 6 years?

1.2.1.4 Applications of Indices in Real-Life situations

Example 1.2.16 Kakamega town's population doubles every 10 years. If the population today is 50,000 people, what will it's population be in 30 years time?

Solution. Since the population doubles every 10 years, we observe the following:

- After 10 years $\rightarrow 50,000 \times 2 = 100,000$ people
- After 20 years $\rightarrow 100,000 \times 2 = 200,000$ people
- After 30 years $\rightarrow 200,000 \times 2 = 400,000$ people

Instead of calculating step by step, we can use indices.

Since the population doubles every 10 years, we use the exponential growth model:

$$P = P_0 \times P^{\frac{t}{10}}$$

Where:

- P = population after t years
- P_0 = Initial population (50000)
- t = number of years (30) years
- The base 2 represents doubling every 10 years

Therefore:

$$P = 50000 \times 2^{\frac{30}{10}}$$

Since $\frac{30}{10} = 3$, we simplify:

$$P = 50000 \times 2^3$$

We now calculate 2^3 ;

$$2^3 = 2 \times 2 \times 2 = 8$$

Now,

$$P = 50,000 \times 8$$

$$= 400,000$$

In 30 years time, the town's population will be 400,000 people □

Checkpoint 1.2.17 Population Growth Model Using Indices. Load the question by clicking the button below.A small community in rural Kenya has an initial population of 1050 people. The population grows each year by a factor of $\frac{53}{50}$. Using indices, calculate the population after 3 years.

Write your answer using the model:

$$P(t) = P_0 \times g^t$$

Answer. $\frac{3126417}{2500}$

Solution. Worked Solution The population model is:

$$P(t) = 1050 \times (\frac{53}{50})^3.$$

Evaluating this gives:

$$P(t) = \frac{3126417}{2500}.$$

Checkpoint 1.2.18 Application of Indices in Compound Growth. Load the question by clicking the button below.A school STEM club invests Ksh 1100 in a small project account. The money grows by a factor of $\frac{11}{10}$ each year. Using indices, calculate the value of the investment after 3 years.

Give your answer in the form:

$$A = P \times r^n$$

Answer. $\frac{14641}{10}$

Solution. Worked Solution The value after 3 years with growth factor $\frac{11}{10}$ is given by:

$$A = 1100 \times (\frac{11}{10})^3.$$

This evaluates to:

$$A = \frac{14641}{10}.$$

Exercises

1. The Science Club of Bidii Secondary School is conducting an experiment on bacteria growth. They place $1,000$ bacteria in a petri dish and observe that the bacteria triple every 5 hours.

a) How many bacteria will be present after 10 hours?

b) How many bacteria will be present after 20 hours?

c) How long will it take for the bacteria to reach $243,000$?

2. During the tree planting month in the school, the principal planted 3 trees in the first week. Each week, the number of trees planted by the principle triples.

a) How many trees will be planted by the 5^{th} week?

b) How long will it take for the principal to plant at least $2,000$ trees?

3. Upendo Bank offers compound interest where an investment grows by a factor of 1.05 per year. A person invests Ksh $50,000$.

a) Write an index notation for the amount after 10 years.

b) Find the total amount after 10 years.

4. The storage capacity of computers has been increasing exponentially. The school library computer storage was 2 GB in the year 2000, and its capacity doubles every 2 years.

a) Write an expression using indices for the capacity after 8 years.

b) What will the storage capacity be after 5 years?

5. A doctor prescribes a medicine that reduces to $\frac{1}{4}$ of its original amount in the body every 6 hours.

a) Express the remaining amount after 18 hours in index form.

b) If the initial dosage was 200 mg, calculate the amount left after 18 hours.

6. The intensity of sound is measured in decibels and follows an exponential scale. If a normal conversation is 10^2 times louder than a whisper and a jet engine is 10 times louder than a whisper:

a) How many times louder is a jet engine compared to a normal conversation?

b) Express this in index notation.

1.2.2 Logarithms

Why Logarithms? In the previous section, indices allowed us to simplify very large or small numbers by expressing them in terms of powers. However, there are situations where we need to reverse this process, rather than finding the result of a power, we need to determine the exponent itself. This is where logarithms come into play.

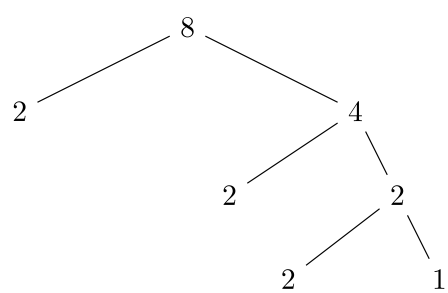
A *logarithm* is the inverse of an exponent (index). It tells us the power to which a specific base must be raised to produce a given number.

For example:
Exponent Form: $2^3 = 8$.
Logarithm Form: $log_2 8 = 3$.
In simpler terms, a *logarithm* answers the question: "To what power must the base be raised to produce a certain number?"
Why Are Logarithms Important?
Logarithms are crucial for solving equations involving exponents. They simplify computations and are widely used in fields like:

- **Finance:** calculating compound interest or investment growth.
- **Science and Engineering:** understanding radioactive decay, measuring sound intensity, or modeling population growth.

1.2.2.1 Logarithms notation

Activity 1.2.5 Work in groups: Form groups of 2 or 3 students
Materials: A Paper/book and a pen
Instructions:
Pick a number from the set: $81, 243, 512$ or 1000
Break down the chosen number into its prime factors.
Start by dividing the number by its smallest prime factor repeatedly until only prime factors remain.
e.g. $8 = 2 \times 2 \times 2 = 2^3$



Express your final result in index form
Reflect on how the indices relate to logarithms:
For example, $2^3 = 8$ translates to $\log_2 8 = 3$.
Key Takeaway:
Logarithms are simply another way of expressing indices. They bridge the gap between exponential and logarithmic notation:

- Exponential Form: $a^b = c$.

  Logarithmic Form:$\log_a c = b$.

e.g

$$8 = 2 \times 2 \times 2 = 2^3$$

$$4 = 2 \times 2 = 2^2$$

$$2 = 2^1$$

The powers/ indices $3, 2, 1$ are the *logarithms*
For $2^3 = 8$ is written as $log_2 8 = 3$

And is read as: logarithm of 8 to base 2 is equal to 3
The general form is:

$$a^b = c \; log_a c = b$$

$a^b = c$ represent the Index notation
$log_a c = b$ represent the Logarithmic notation

Example 1.2.19 The table below contains numbers in **index form** and **logarithm form**. Fill and complete the table.

Table 1.2.20

Exponential Form	**Logarithmic Form**
___	$\log_2 32 = 5$
$10^4 = 10,000$	___
$3^3 = 27$	___
$5^2 = 25$	___
___	$\log_7 7 = 1$
___	$\log_4 64 = 3$

□

Example 1.2.21 What is the logarithmic form of the following exponents:

a) $6^2 = 36$
When $6^2 = 36$ then, $\log_6 36 = 2$

b) $9^3 = 729$
When $9^3 = 729$ then, $\log_9 729 = 3$

c) $4^5 = 1024$
When $4^5 = 1024$ then, $\log_4 1024 = 5$

□

Checkpoint 1.2.22 Compute Logs Base 10. Load the question by clicking the button below.Evaluate the following.

1. $\log_{10}(10) =$ _____
2. $\log_{10}(0.001) =$ _____
3. $\log_{10}\left(10^{\frac{2}{5}}\right) =$ _____

Answer 1. 1
Answer 2. -3
Answer 3. $\frac{2}{5}$
Solution. Since $\log_a A^k = k \log_a A$, we have
$\log_{10}(10) = \log_{10} 10^1 = 1,$
$\log_{10}(0.001) = \log_{10} 10^{-3} = -3,$
$\log_{10}\left(10^{\frac{2}{5}}\right) = \dfrac{2}{5}.$

Checkpoint 1.2.23 Expressing an Exponential Number in Logarithmic Form. Load the question by clicking the button below.A computing device stores data in powers of a specific base. In one task, it processes a total of 1000 units using base 10.

Write the logarithmic form of the statement:

$$10^3 = 1000$$

Express your answer as: $\log_a(b)$.

Answer. $\frac{\ln(1000)}{\ln(10)}$

Solution. Worked Solution The exponential form is:

$$10^3 = 1000.$$

To convert from exponential to logarithmic form, we use:

$$a^b = c \quad \longrightarrow \quad \log_a(c) = b.$$

Therefore:

$$\log_{10}(1000) = 3.$$

Checkpoint 1.2.24 Rewrite Complex Exponential Relationship into Logarithmic Form. Load the question by clicking the button below.During a simulation, a machine amplifies a signal using exponential growth. After several cycles, the signal reaches a strength of 64, generated using base 4.

Write the logarithmic form of the statement:

$$4^3 = 64$$

Express your answer in the form $\log_a(b)$.

Answer. $\frac{\ln(64)}{\ln(4)}$

Solution. Worked Solution The exponential relationship is:

$$4^3 = 64.$$

We use the identity:

$$a^k = N \quad \Longleftrightarrow \quad \log_a(N) = k.$$

So:

$$\log_4(64) = 3.$$

This is written as:

$$\frac{\log(64)}{\log(4)}.$$

Exercises

1. What are the logarithmic form of:

a) $2^6 = 64$

b) $5^3 = 125$

c) $3^x = 81$

e) $8^4 = 4096$

f) $6^y = 216$

g) $9^{\frac{1}{2}} = 3$

d) $8^{\frac{2}{3}} = 4$ h) $4^{-2} = \frac{1}{16}$

2. Express $10^4 = 10000$ in logarithmic form.

3. Find the value of y given that $\log_y 81 = 4$

4. Solve for x if $\log_2 x = 5$

5. Convert $8^x = 512$ to logarithmic form and solve for x.

1.2.2.2 Using Logarithms with Standard Form

Standard form is a way of writing very large or very small numbers in a more manageable format. It is expressed as:

$$B \times 10^x$$

Where:

- B is a number between 1 and 10 ($1 \leq B < 100$)
- x is an integer (positive for large numbers and negative for small numbers)

Activity 1.2.6 Work in groups: Form groups of 2 or 3 students
Instruction:

a) Write any five digit number in your book

b) Identify the first non-zero digit from the left.

c) Place a decimal point after this digit.

d) Count how many places the decimal has moved;

- If moved left, the exponent is positive.
- If moved right, the exponent is negative.

e) Express the number in the form $B \times 10^x$

Examples:

- The distance from the Earth to the Sun is approximately $149,600,000\,km$.

$$149,600,000 = 1.496 \times 10^8\,km$$

- The size of a red blood cell is about 0.000007 m

$$0.000007 = 7.0 \times 10^{-6}m$$

Further Activity:

- The population of Kenya is approximately $54,985,698$. Express this number in standard form.
- The size of a human hair is about 0.00008 metres. Express the length in standard form.
- The charge of an electron is 0.00000000000000000016 coulombs. Convert this to standard form.

Finding Logarithms of Numbers in Standard Form:

Example 1.2.25 Find $\log(4.5 \times 10^5)$

Solution.

$$\log(4.5 \times 10^5) = \log 4.5 \times 10^5$$

$$\log 4.5 = 0.6535$$

$$\therefore \log 450000 = 5.6535$$

□

In this case 5 is the **characteristic**. The characteristic is the whole number part of a logarithm. It tells us how large or small the number is based on powers of 10.

0.6535 is the **mantissa**. The mantissa is the decimal part of the logarithm, found using logarithm tables. It depends on the significant digits of the number but is always positive.

Key Takeaway:

- For numbers greater than 1, the characteristic is one less than the number of digits before the decimal point.
- For numbers less than 1, the characteristic is negative, often written in bar notation (e.g $\overline{3}$ instead of -3).

Checkpoint 1.2.26 Using Logarithms with Standard Form. Load the question by clicking the button below.Write the number 500000 in standard form using logarithms.

Express your answer in the form $A \times 10^B$.

Answer. 500000

Solution. Worked Solution We want to express the number in standard form $A \times 10^B$.

$500000 = 5 \times 10^5$.

Using logarithms:

$$\log_{10}(500000) = \log_{10}(5) + \log_{10}(10^5)$$

$$= \log_{10}(5) + 5.$$

Therefore the standard form is:

$$5 \times 10^5.$$

Checkpoint 1.2.27 Using Logarithms to Find the Power Part of Standard Form. Load the question by clicking the button below.A scientific calculator displays a number as $1.4e+8$ Use logarithms to determine its standard form in the form: $A \times 10^B$

Answer. $1.4e+8$

Solution. Worked Solution We use logarithms to find the exponent B.

$$\log_{10}(1.4e+8) = \log_{10}(1.4) + 8$$

Thus:

$$A = 1.4, \quad B = 8.$$

Therefore the standard form is:

$$1.4 \times 10^8.$$

1.2.2.3 Determining Common Logarithms Using Mathematical Tables and Calculators

Common Logarithms

Common logarithms, also known as logarithms to base 10, are written as $\log x$ instead of $\log_{10} x$.

When

$$10x = y$$

It means that 10 must be raised to the power of y to give x, that is:

$$10^y = x$$

For example: $\log 100 = 2$ because $10^2 = 100$

Common logarithms simplify complex multiplications, divisions, and exponentiation by converting them into easier operations using logarithmic laws.

Using Logarithm Tables

In Grade 8, you learned how to use mathematical tables to find squares, square roots, and reciprocals of numbers. Here we will use the mathematical tables to determine logarithms to base 10.

Here is an extract of a logarithm table to base 10

											ADD								
x	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
1.0	.0000	0043	0086	0128	0170	0212					4	8	13	17	21	25	29	34	38
						0212	0253	0294	0334	0374	4	8	12	16	20	24	28	32	36
1.1	.0414	0453	0492	0531	0569	0607					4	8	12	16	19	23	27	31	35
						0607	0645	0682	0719	0755	4	7	11	15	19	22	26	30	33
1.2	.0792	0828	0864	0899	0934	0969					4	7	11	14	18	21	25	28	32
						0969	1004	1038	1072	1106	3	7	10	14	17	20	24	27	31
1.3	.1139	1173	1206	1239	1271	1303					3	7	10	13	16	20	23	26	30
						1303	1335	1367	1399	1430	3	6	10	13	16	19	22	26	29
1.4	.1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	27
1.5	.1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14	17	20	22	25
1.6	.2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	10	13	16	18	21	23
1.7	.2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
1.8	.2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	10	12	14	17	19	22
1.9	.2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11	13	15	18	20
2.0	.3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
2.1	.3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
2.2	.3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	11	13	15	17
2.3	.3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	5	7	9	11	13	14	16
2.4	.3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	13	14	16
2.5	.3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
2.6	.4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	6	8	10	11	13	14
2.7	.4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	10	11	13	14
2.8	.4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
2.9	.4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
3.0	.4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	8	10	12	13
3.1	.4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	13
3.2	.5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	10	12
3.3	.5185	5198	5211	5224	5237	5250	.5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
3.4	.5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	12
3.5	.5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	8	10	11
3.6	.5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
3.7	.5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	4	5	6	7	8	10	11
3.8	.5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	4	6	7	8	9	10
3.9	.5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	6	7	8	9	10
4.0	.6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	7	8	9	10
4.1	.6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
4.2	.6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
4.3	.6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
4.4	.6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
4.5	.6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
4.6	.6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	5	6	7	8
4.7	.6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
4.8	.6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
4.9	.6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
5.0	.6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	4	4	5	6	7	8
5.1	.7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	2	3	4	5	6	6	7
5.2	.7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	6	7
5.3	.7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9

Log π = 0.4972

Positive Numbers greater than 1

Example 1.2.28

1. Find log(472.8) using a logarithm table.

Solution. Using a logarithm table, follow these steps:

1. Identify the characteristic:
   Write 472.8 in standard form

$$\log(472.8) = \log(4.728 \times 10^2)$$

The characteristic is 2 since $10^2 \leq 472.8 < 10^3$.

2. Find the mantissa from the log table:

   Locate 47 in the first column and move to the column labeled 2.

   Read the value: 0.6749.

$$\log(472) = 0.6749$$

3. Apply the mean difference for 8:

   From the table, the mean difference for 8 is 0.0007.

   Add this to the mantissa:

$$0.6749 + 0.0007 = 0.6756$$

4. Combine with the characteristic

   Therefore;

$$\log(472.8) = 2.6756$$

Using a calculator, we can confirm that $\log(472.8) \approx 2.6756$, matching the value obtained from the logarithm table. □

Positive Numbers less than 1

Example 1.2.29 Use logarithm tables to solve $\log 0.00534$

Solution. Since 0.00534 cannot be read directly from the table, first we need to write it in standard form;

Standard form of

$$0.00534 = 5.34 \times 10^{-3}$$

Here, the characteristic is -3 and the mantissa is the logarithm of 5.34. Using the logarithm table:

- Locate 53 in the first column.
- Move to the column labeled 4.
- Read the value: $\log(5.34) = 0.7272$.

Now, apply the logarithm property:

$$\log(0.00534) = \log(5.34) + \log(10^{-3})$$

Since $\log(10^{-3}) = -3$, we substitute:

$$\log(0.00534) = 0.7272 - 3$$

$$= \bar{3}.7272$$

The bar over 3 represents a negative characteristic.

$$\therefore \log 0.00534 = \bar{3}.7272$$

□

Exercises

1. Express the following numbers in standard form.

a) 4820
b) 37.6
c) 672000
d) 321000
e) 0.000485
f) 91800
g) 5.27×10^5
h) 0.000672

2. Use a logarithm table to find the logarithm of the following numbers:

a) 0.00893
b) 3140
c) 52.7
d) 0.000978
e) 0.000245
f) 6.42×10^3
g) 78900

Activity 1.2.7 Exploring Logarithms in Real-Life Applications. At individual level, go explore how logarithms are used in different fields such as engineering, finance, and science.

1. Research and describe two real-life applications of logarithms. Present your findings to the class.
2. The pH scale in chemistry is based on logarithms. Given that pH is defined as pH $= -\log[H^+]$, determine the pH of a solution where $[H^+] = 3.2 \times 10^{-4}$.
3. The Richter scale measures earthquake intensity using the formula $R = \log\left(\frac{I}{I_0}\right)$, where I is the intensity of the earthquake and I_0 is the reference intensity. If an earthquake is 1000 times more intense than the reference, what is its magnitude on the Richter scale?
4. Use logarithm tables to evaluate $\sqrt{\frac{(6.28\times42.5)}{9.81}}$.
5. A country's population grows exponentially according to the formula $P = P_0e^{rt}$, where P_0 is the initial population, r is the growth rate, and t is time in years. Solve for t if a population doubles in 10 years at a growth rate of 5% per year.

Antilogarithms

The antilogarithm (antilog) of a number is the inverse operation of taking a logarithm. If $\log x = y$, then x is the antilog of y. This means:

$$x = 10^y$$

To find the antilogarithm of a number, we use logarithm tables or a calculator.

Finding Antilogarithms Using Logarithm Tables

To find the antilogarithm of a given logarithmic value using a logarithm table, follow these steps:

- Separate the characteristic and mantissa.
- Use the logarithm table to find the antilog of the mantissa.

- Apply the characteristic as a power of 10.

Example 1.2.30 Find the antilogarithm of 2.6756 using logarithm tables.

Solution. Separate the characteristic and mantissa

$$2.6756 = 2 + 0.6756$$

Find the antilogarithm of 0.6756 using tables:

$$\text{Antilog }(0.6756) = 4.735$$

Apply the characteristic:

$$10^2 \times 4.735 = 473.5$$

Thus, the antilogarithm of 2.6756 is 473.5. □

Finding Antilogarithms Using a Calculator

When using a calculator, simply input the given logarithm value and use the inverse logarithm function:

$$x = 10^y$$

For example, to find Antilog (1.3562) on a calculator:

- Press [10^x] or [INV] + [LOG] and enter 1.3562.
- Record the results

Checkpoint 1.2.31 Computing Simple Logarithms. Load the question by clicking the button below.Find the values of the following.

1. $\log_5 625 =$ _____
2. $5^{\log_5 22} =$ _____

Answer 1. 4

Answer 2. 22

Solution.

1. $\log_5 625 = \log_5 5^4 = 4$.
2. Since raising to the power and taking logs are inverse operations (previous key point), we have:
   $5^{\log_5 22} = 22$.

Checkpoint 1.2.32 Determining Common Logarithms Using a Calculator. Load the question by clicking the button below.

An error occurred while processing this question.

Exercises

1. Find the antilogarithms of the following logarithmic values using logarithm tables:

i) $\log P = 1.3562$

ii) $\log Q = 3.4821$

iii) $\log R = 0.7294$

iv) $\log S = 4.2187$

2. Solve the following logarithms using a calculator:

i) $\log a = 2.1457$

ii) $\log b = 0.8743$

iii) $\log c = 3.5961$

iv) $\log d = 1.9999$

1.2.2.4 Multiplying and Dividing Logarithms

Logarithms simplify multiplication and division by converting them into addition and subtraction, respectively. This is particularly useful when dealing with large numbers.

Example 1.2.33 Find 236.5×42.8 using logarithm table.

Solution. Find the logarithm of each number from the logarithm table:

$$\log 236.5 = 2.3741, \quad \log 42.8 = 1.6318$$

Add the logarithms:

$$2.3741 + 1.6318 = 4.0059$$

Find the antilogarithm of 4.0059:

$$\text{Antilog}(4.0059) \approx 10160$$

Thus, $236.5 \times 42.8 \approx 10160$. □

Example 1.2.34 Find $\frac{528.6}{24.7}$ using logarithm tables.

Solution. Find the logarithm of each number from the logarithm table:

$$\log 528.6 = 2.7233, \quad \log 24.7 = 1.3927$$

Subtract the logarithms:

$$2.7233 - 1.3927 = 1.3306$$

Find the antilogarithm of 1.3306:

$$\text{Antilog}(1.3306) \approx 21.4$$

Thus, $\frac{528.6}{24.7} \approx 21.4$. □

1.2.2.5 Applying Logarithms to Powers and Roots

Logarithms can be used to simplify calculations involving powers and roots by converting exponentiation into multiplication and roots into division.

Activity 1.2.8 Material needed:

- Pen and paper
- A scientific calculator

Instructions:
Use your calculator and math reasoning to solve the following expressions involving logarithms:

$$\log(10000)$$

Now rewrite 10000 as a power of 10:

$$\log(10000) = \log(10^4)$$

Apply the logarithmic law:

$$\log(a^b) = b \times \log(a)$$

$$\log(10^4) = 4 \times \log(10)$$

$$= 4 \times 1 = 4$$

Try the following:

- $$\log(100)$$
- $$\log(1000000)$$
- $$\log(\sqrt{100})$$
- $$\log(\sqrt[3]{1000})$$
- What do you observe when applying logarithms to square and cube roots?
- Why does
  $$\log(\sqrt{100})$$
  give half of
  $$\log(100)?$$
- What general rule can you form for logarithms and powers?

Key Takeaway:
Logarithms help simplify calculations involving powers and roots.
For example: $\log(a^b) = b \times \log(a)$, and $\log(\sqrt{a}) = \frac{1}{2} \times \log(a)$.

Example 1.2.35 Evaluate $(23.5)^4$ using logarithm tables.

Solution. Find the logarithm of 23.5 from the logarithm table:

$$\log 23.5 = 1.3711$$

Multiply by the exponent 4:

$$1.3711 \times 4 = 5.4844$$

Find the antilogarithm of 5.4844:

$$\text{Antilog}(5.4844) \approx 304000$$

Thus, $(23.5)^4 \approx 304000$. □

Example 1.2.36 Evaluate $\sqrt[3]{524.8}$ using logarithm tables.

Solution. Find the logarithm of 524.8 from the logarithm table:

$$\log 524.8 = 2.7200$$

Divide the logarithm by 3 (since it is a cube root):

$$\frac{2.7200}{3} = 0.9067$$

Find the antilogarithm of 0.9067:

$$\text{Antilog}(0.9067) \approx 8.1$$

Thus, $\sqrt[3]{524.8} \approx 8.1$. □

Exercises

1. Use logarithm tables to evaluate the following:

a) 345.6×78.9

b) 6284×92.5

c) 0.0482×53.7

2. Use logarithms to calculate:

a) $652.3 \div 12.7$

b) $0.0854 \div 3.42$

c) $4312 \div 58.3$

3. Solve the following using logarithms:

a) $(78.5)^3$

b) $(254.6)^4$

c) $(12.75)^{2.5}$

4. Compute the following roots using logarithms:

a) $\sqrt[3]{658.4}$

b) $\sqrt{82.6}$

c) $\sqrt[4]{3126}$

5. A square field has an area of 18432 square metres. Use logarithms to determine the length of one side.

6. Use logarithm tables to evaluate:

a) $\sqrt{\frac{(4.56 \times 12.3)}{24.7}}$

b) $\sqrt{\frac{(2.718 \times 9.81)}{5.432 \times 3.14}}$

c) $\frac{\sqrt{52.8 \times 24.6}}{\sqrt{31.5}}$

d) $\sqrt[3]{\dfrac{(6.75 \times 432)}{0.89}}$

e) $\sqrt[4]{\dfrac{(8462 \times 23.7)}{673}}$

7. The volume of a cube is 79507 cubic centimetres. Use logarithms to find the length of one side.

1.3 Quadratic Expressions and Equations 1

In this section, we will explore quadratic expressions and equations, which are important concepts in algebra. Quadratics might sound complicated, but they are actually all around us, and learning how to work with them will make solving many kinds of problems much easier. We will start by understanding what quadratic expressions and equations are and how they relate to real-life situations, like calculating areas. You will also learn different ways to solve these equations, such as factorization. By the end, you will not only know how to solve quadratic equations but also understand how they apply to everyday problems.

- Define quadratic expressions and equations.
- Derive quadratic identities from the concept of area.
- Solve quadratic equations by factorization.
- Derive quadratic formula and use it to solve quadratic equations.
- Form quadratic equations different in different situations.
- Explore use of quadratic equations in real life situations.

1.3.1 Quadratic Expressions

Activity 1.3.1 ***Work in Groups.***

- Identify and label the terms of the quadratic expression: **quadratic term, linear term**, and **constant term**.
- Discuss whether the expression is in standard form and, if not, try to convert it.
- Solve the quadratic equation.
- Discuss the properties of the expression: whether it can be solved, whether it has real solutions, etc.
- Examples of quadratic expressions:
  - $2x^2 + 5x - 3$
  - $3x^2 - 4x + 1$
- Identify the coefficients and constants.
- Present your findings to the class.

By the end of this subsection, we will learn how to form quadratic expressions from different scenarios, simplify them, and use them in various real-life applications. By the end, you should be able to recognize quadratic expressions in different situations and form them based on the given conditions.

A quadratic expression is an expression of the form:

$$ax^2 + bx + c$$

where;

- a, b and c are constants (real numbers).
- x is a variable (what we look for or finding), and
- $a \neq 0$ beacuse if $a = 0$, it would not be a quadratic expression.

In the previous grades you learned about algebraic expressions

An expression of the form;

$$3x \text{ or } y^2$$

is called a **monomial expression**

A **monomial expression** is a an expression with one term.

A **binomial expression** is an expression with two terms *ie.*

$$(a + b) \text{ or } (c + d)$$

Expressions can be formed in various ways;

- Multiplying an expresssion with one term (*monmial*) with an expression with two terms (*a binomial*)
- Multiplying two expression with two terms (*two binomials*).

1.3.1.1 Multiplying a monomial by a binomial.

Activity 1.3.2 ***Work in Groups***

The following data represents the multiplication of a monomial by a binomial. Copy the table and complete it.

$3x$	$(x + 4)$	$3x(x + 4)$	$3x^2 + 12x$
$2y$	$(y - 5)$	$2y(y - 5)$	
$-4a$	$(a + 7)$	$-4a(a + 7)$	
$6b$	$(b - 3)$	$6b(b - 3)$	

Multiply the monomial by each binomial using the distributive property. For example, for $3x(x + 4)$, distribute $3x$ across the binomial and simplify.

1. What do you notice about the process of multiplying a monomial by a binomial?
2. How do you apply the distributive property when multiplying terms?
3. Can you create your own examples of a monomial multiplied by a binomial and solve them?
4. How can you simplify the expressions after distributing the monomial?
5. Discuss in your group how the distributive property allows you to break down the multiplication step-by-step.

Key Takeaway.

When multiplying a monomial by a binomial, you apply the distributive property, multiplying the monomial by each term in the binomial, then combining like terms if necessary.

Example 1.3.1
Simplify:

$$2a(a-1)-3(a^2-1)$$

Solution. Opening the brackets

$$2a(a-1)-3(a^2-1)=2a(a)+2a(-1)+(-3)(a^2)+(-3)(-1)$$

$$=2a^2-2a-3a^2+3$$

Collecting like terms we form a quadratic expression:

$$-a^2-2a+3$$

□

From the example above you can see that;
$(2a)$ and(-3) are the **monomial expressions** while $(a-1)$ and(a^2-1) are the **binomial expression.**

Example 1.3.2
Simplify the following expressions to form a quadratic expression.

$$5(2x^2+5)+6x(x-2)$$

Solution. Opening the brackets:

$$5(2x^2+5)+6x(x-2)=10x^2+25+6x^2-12x$$

Collecting like terms:

$$6x^2+10x^2-12x+25$$

Simplifying:

$$16x^2-12x+25$$

□

Exercises

1. Form a quadratic equation using the following terms:

(i) $3x(x+2)-4(x^2-1)$

(ii) $8n(n+5)-3(n^2-6)$

(iii) $4p(p-1)-5(p^2+2)$

(iv) $7a(a+3)-2(a^2-2)$

2.

(a) $3m(m-2)-6(m^2+1)$

(b) $9x(x+3)-4(x^2-4)$

(c) $2y(y-1)-3(y^2+2)$

(d) $2u^2-2(2u+9)$

3.

(i) $5a(a-3)-2(a^2+4)$

(ii) $9x(x-4)-1$

(iii) $13-(x+4)^2$

(iv) $p(-2p)+2(p-1)$

4.

(a) $2b(b+4)-3(b^2-2)$

(b) $5q(q+4)-2(q^2-3)$

(c) $2r(r+5)-r^2+7$

(d) $2m-2(m-1)^2$

1.3.1.2 Multipling Two Binomials.

Activity 1.3.3 ***Work in Groups***

In groups of 3, look through the table below. The following pairs of expressions are given. Complete the table by multiplying the two binomials in the first two columns and writing the result in the third and fourth column.

$(x+3)$	$(x+5)$	(x + 3)(x + 5)	$x^2+8x+15$
$(y-4)$	$(y+2)$	$(y-4)(y+2)$	y^2-2y-8
$(a+6)$	$(a-1)$		
$(b-2)$	$(b+7)$		

(a) After you have completed the table, discuss with your group members and write down the results in the table below.

(b) Discuss the pattern of the terms in the expanded form when both binomials have positive or negative terms.

(c) What do you notice about the results?

(d) How do the signs in the expressions affect th final expanded expression?

(e) Can you create and expand your own binomial multiplication problems?

Investigation 1.3.4

What is the FOIL method, and why is it useful for multiplying binomials?

Suppose we want to Multiply

$$(a+b)(c+d)$$

We have;

$$(a+b)(c+d)=(ac)+(ad)+(bc)+(bd)$$

Example 1.3.3

Find the product:

$$(3x-2)(5x+8)$$

Solution.

$$(3x)\cdot(5x)=15x^2$$
$$(3x)\cdot(8)=24x$$
$$(-2)\cdot(5x)=-10x$$
$$(-2)\cdot(8)=-16$$

Now combine the terms:

$$15x^2 + 24x - 10x + 16 = 15x^2 + 14x - 16$$

Therefore;

$$(3x - 2)(5x + 8) = 15x^2 + 14x - 16$$

□

From the example we can see we formed an algrebaric expressions

The product of two identical binomials is known as the square of the binomial and is written as:

$$(a + b)^2$$

$$(a + b)^2 = a^2 + ab + ab + b^2$$

$$= a^2 + 2ab + b^2$$

Therefore;

$$(a + b)^2 = a^2 + 2ab + b^2$$

If the two terms are of the form (ax + b) and (ax - b) then their product is:

$$(a + b)(a - b)$$

$$(a + b)(a - b) = a^2 - ab + ab - b^2$$

Collecting terms;

$$= a^2 - b^2$$

Thus, the product is:

$$(a + b)(a - b)a^2 - b^2$$

Now this is the difference of squares formula.

Example 1.3.4

Form a qudratic expression:

$$(8x + 5)^2$$

Solution.

$$(8x + 5)^2 = (8x + 5)(8x + 5)$$

Expanding the brackets

$$64x^2 + 40x + 40x + 25$$

Collect like terms and form a quadratic expressions:

$$64x^2 + 80x + 25$$

□

Exercises

1. Form quadratic expression:

(a). $2y(y+4)$
(b). $3x(2x+5)$
(c). $-4y(3y-2)$
(d). $-3x(4x+7)$

2.

(a). $(y+5)(y+2)$
(b). $-(4-x)(x+4)$
(c). $(s+6)^2$
(d). $\left(\left(\frac{1}{4}-\frac{1}{x}\right)^2\right)^2$
(e). $(1-3h)(1+3h)$
(f). $(2p+3)(2p+2)$
(g). $(-10)(2y2+8y+3)$
(h). $\left(x+\frac{4}{x}\right)^2$

3.

(a) $(x-1)(x-6)$
(b) $(x-2)(x-3)$
(c) $(2x+3)(x+4)$
(d) $(3y-5)(2y+7)$

1.3.2 Quadratic Identities.

Activity 1.3.5 ***Work in Groups.***
Form a group of atleast 4. individuals. Define, discuss and work on the following:

1. Quadratic identities.
2. Difference of squares.
3. Perfect squares.
4. Factorization of quadratic expressions.

Copy the following expressions and identies, observe and discuss:

(i). $(a+b)^2 = a^2 + 2ab + b^2$

(ii). $(a-b)^2 = a^2 - 2ab + b^2$

(iii). $(a-b)(a+b) = a^2 - b^2$ (Difference of squares)

- Compare the different approaches groups used to solve similar problems.
- Discuss how quadratic identities can make simplification and factoring easier.
- Explore how these identities are useful in different contexts (e.g., solving quadratic equations, simplifying expressions in algebra).
- How do the identities help us solve quadratic expressions faster?
- What happens if we don't recognize the identity right away—how might that slow us down?
- Can you think of any real-world applications where you might use quadratic identities?s

Suppose we have a equation P and Q.
In mathematics, an equation $P = Q$ is called an ***identity*** if the following

conditions are satisfied:

1. Both sides of the equality relation contain some variables.

2. Both the sides give the same value when the variable is substituted with a particular constant.

Note 1.3.5

If an equation satisfies the two conditions mentioned above, it is known as an identity. There are many mathematical relations that can be classified as identities, but it's not necessary to memorize all of them. However, certain key identities in algebra are essential, as they simplify calculations.

Quadratic identities are special rules or formulas that help us work with ***quadratic equations***.

Common quadratic identities are:

- Difference of squares.

- Perfect squares.

1.3.2.1 Difference of Squares.

Activity 1.3.6 ***Work in Groups.***

Define, discuss, and work on the following:

1. Difference of squares.

2. Identifying difference of squares in algebraic expressions.

3. Factoring using the difference of squares formula.

4. Recognizing patterns when applying the difference of squares.

Copy the following expressions and identities, observe and discuss:

(i). $a^2 - b^2 = (a + b)(a - b)$ (Difference of squares)

Recognize how the difference of squares applies to algebraic expressions.

- Compare the different approaches groups used to solve similar problems.

- Discuss how the difference of squares formula makes factoring easier.

- Explore how this identity is useful in different contexts (e.g., simplifying expressions, solving equations).

- How does recognizing the difference of squares help in solving quadratic equations?

- What happens if we don't recognize the difference of squares—how might that affect solving the problem?

- Can you think of any real-world applications where the difference of squares might be used?

The ***Difference of Squares*** is an identity that applies when you have two terms that are perfect squares and are being subtracted from each other. The identity states:

$$a^2 - b^2 = (a - b)(a + b)$$

This means that the difference between two squares can be factored into the product of two binomials: one where the terms are subtracted and one where the terms are added.

In the expression $a^2 - b^2$, a and b represent the square roots of the two terms. When you subtract two perfect squares, you can factor the expression as the product of two binomials: $(a - b)$ and $(a + b)$.

Example 1.3.6

Consider the quadratic, solve:

$$x^2 - 16$$

Solution. The expression $x^2 - 16$ is in the form of a difference of squares, which follows the identity:

$$a^2 - b^2 = (a - b)(a + b)$$

In this case, $a^2 = x^2$ and $b^2 = 16$. The square roots of x^2 and 16 are x and 4, respectively. We can rewrite the expression as:

$$x^2 - 16 = x^2 - 4^2$$

Applying the difference of squares identity we have:

$$x^2 - 16 = (x - 4)(x + 4)$$

□

1.3.2.2 Perfect Squares.

Activity 1.3.7 *Work in Groups.*

Define, discuss, and work on the following:

1. Perfect square identities.
2. Expanding perfect squares.
3. Recognizing perfect square trinomials.
4. Factoring perfect square trinomials.

Copy the following expressions and identities, observe and discuss:

(i). $(a + b)^2 = a^2 + 2ab + b^2$

(ii). $(a - b)^2 = a^2 - 2ab + b^2$

Recognize the middle term is twice the product of the two terms.

- Compare the different approaches groups used to solve similar problems.
- Discuss how perfect square identities can make simplification and factoring easier.
- Explore how these identities are useful in different contexts (e.g., solving quadratic equations, simplifying expressions in algebra).
- How do perfect square identities help us solve quadratic expressions faster?

- What happens if we don't recognize the identity right away—how might that slow us down?
- Can you think of any real-world applications where you might use perfect square identities?

A ***Perfect Square*** is a special kind of trinomial that can be factored into the square of a binomial. There are two forms of perfect square identities:

$$a^2 + 2ab + b^2 = (a+b)^2 \quad \textit{and} \quad a^2 - 2ab + b^2 = (a-b)^2$$

In these identities, the expression on the left is a perfect square, which means it can be written as the square of a binomial (a two-term expression). The first identity is used when the middle term is positive, and the second identity is used when the middle term is negative.

Example 1.3.7 (Using the first identity).
Consider the quadratic below:

$$x^2 + 6x + 9$$

Solution. The expression $x^2 + 6x + 9$ is perfefect square. A perfect square follows the pattern:

$$(a+b)^2 = a^2 + 2ab + b^2$$

Here, $a^2 = x^2, 2ab = 6x$ and $b^2 = 9$. The square roots of x^2 and 9 are x and 3, respectively.

In this case, the expression $x^2 + 6x + 9$ matches the form $a^2 + 2ab + b^2$, where:

- $a = x$,
- $b = 3$.

The middle term $2ab$ is $6x$, which fits the formula for a perfect square with a negative middle term. Applying the identity:

$$x^2 + 6x + 9 = (x+3)^2$$

So, $x^2 + 6x + 9$ factors into $(x+3)^2$. □

Example 1.3.8 (Using the second identity).
Using the quadratic below:

$$x^2 - 10x + 25$$

Solution. The expression $x^2 - 10x + 25$ matches the form $a^2 - 2ab + b^2$ where:

- $a = x$,
- $b = 5$.
- The middle term $2ab = -10x$, and the constant term $b^2 = 25$.

So, this is a perfect square, and we can factor it as:

$$x^2 - 10x + 25 = (x-5)^2$$

□

1.3.2.3 Factoring Quadratic.

Activity 1.3.8 ***Work in Groups.***
Define, discuss, and work on the following:

1. Factorization of quadratic expressions.
2. Identifying common factors in expressions.
3. Factorizing using the method of splitting the middle term.
4. Recognizing the difference between factoring by grouping and simple factoring.

Copy the following expressions, observe and discuss:

(i). Factorize: $x^2 + 5x + 6$

(ii). Factorize: $x^2 - 7x + 12$

(iii). Factorize: $3x^2 - 15x$

(iv). Factorize by grouping: $x^2 + 4x + 3x + 12$

- Compare the different approaches groups used to factor similar expressions.
- Discuss how factoring helps in solving quadratic equations.
- Explore how recognizing common factors and patterns makes factoring easier.
- How does factoring help us simplify algebraic expressions faster?
- What challenges do you face when factoring complex expressions?
- Can you think of any real-world scenarios where factoring is useful (e.g., optimizing areas, engineering problems)?

Factoring quadratic expressions involves expressing a quadratic equation in the form of two binomials. A general quadratic equation looks like:

$$(ax + m)(bx + n)$$

To factor a quadratic, we find two numbers m and n such that:

i. $m \times n = ac$ (the product of a and c),

ii. $m + n = b$ (the coefficient of x)

Once we find m and n, we break the middle term bx into two terms using m and n, then factor by grouping.

Example 1.3.9
Cosider:

$$x^2 + 5x + 6$$

Solution. In this example $a = 1, b = 5$ and $c = 6$. We need two numbers that when multiplied it gives $ac = 1 \times 6 = 6$ and when the the two values are added up they give $b = 5$. These two numbers are 2 and 3.

$$2 \times 3 = 6 \qquad \text{and} \qquad 2 + 3 = 5$$

Now, rewriting the middle term $5x$ as $2x+3x$, and then factor by grouping

$$x^2 + 2x + 3x + 6$$

Grouping the terms:

$$(x^2 + 2x) + (3x + 6)$$

Factoring each groups or movng common factor of each and create brackets:

$$x(x+2) + 3(x+2)$$

Now, factor out the common binomials:

$$(x+2)(x+3)$$

Finally, $x^2 + 5x + 6$ factors to $(x+2)(x+3)$. □

1.3.2.4 Exercises

1. Use the quadratic identities to write down the expansions of each of the following expressions:

(a) $(4x+5)^2$

(b) $\left(\frac{1}{x}+\frac{1}{y}\right)\left(\frac{1}{x}-\frac{1}{y}\right)$

(c) $(8-x)^2$

(d) $(x-7)^2$

(e) $\left(x+\frac{1}{2}\right)^2$

(f) $\left(\frac{1}{4}-\frac{3}{4}b\right)^2$

(g) $(x+2)^2$

(h) $(x+5)^2$

(i) $(x+2)(x+3)$

1.3.3 Factorisation Of Quadratic Expressions.

Factorisation is the process of breaking down a quadratic expression into a product of simpler binomials.

1.3.3.1 Factorisation When The Coefficient Of x^2 Is One

The expression

$$ax^2 + bx + c,$$

where a, b, c are constants and $a \neq 0$, is called a quadratic expression.

In such expressions a is called the coefficient of x^2, b is called the coefficient of x and c is called the constant term.

When the coefficient of x^2 is one, the expression is of the form;

$$x^2 + bx + c.$$

Cosider the following expressions:

(a). $(x+3)(x+4)$. (b). $(x-6)(x-5)$.

Expanding the expressions we get:

$$(x+3)(x+4) = x(x+3) + 4(x+3)$$

$$(x+3)(x+4) = x^2 + 3x + 4x + 12$$

Collecting like terms to get:

$$x^2 + 7x + 12.$$

Expanding the expressions we get:

$$(x-6)(x-5) = x(x-5) - 6(x-5)$$

$$(x-6)(x-5) = x^2 - 5x - 6x + 30$$

Simplify the middle terms:

$$x^2 - 11x + 30.$$

From the above examples, we can see that the expressions $x^2+7x+12$ and $x^2-11x+30$ are formed from the factors of expressions $(x+3)(x+4)$ and $(x-6)(x-5)$ respectively.

The factorised form of the expression x^2+bx+c is $(x+m)(x+n)$ where m and n are the factors of c whose sum is b.

In each case;

i The sum of the constant terms in the factors is equal to the coefficient of x in the expression.

ii The product of the constant terms in the factors is equal to the constant term in the expression.

Example 1.3.10

Factorise the following expression:

$$x^2 + 5x + 6.$$

Solution. In this case, the coefficient of x^2 is one, the coefficient of x is 5 and the constant term is 6.

So the factors will be of the form $(x+m)(x+n)$ where m and n are the factors of 6 whose sum is 5.

In the expression x^2+5x+6, look for two numbers such the numbers a and b such that

$$a + b = 5$$

is coefficient of x and $ab = 6$ is the constant term.

In this case, the numbers are 2 and 3.

$$x^2 + 5x + 6 = x^2 + 2x + 3x + 6.$$

Grouping terms, we get:

$$x^2 + 5x + 6 = x(x+2) + 3(x+2).$$

$$x^2 + 5x + 6 = (x+2)(x+3).$$

Therefore, the factorised form of the expression x^2+5x+6 is $(x+2)(x+3)$.

□

Example 1.3.11

Factorise the following expression:

$$x^2 - 9x + 20.$$

Solution. Look for two numbers such that the numbers a and b such that $a+b=-9$ and $ab=20$.

The numbers are -4 and -5.
Then, the expression $x^2 - 9x + 20$ can be written as:

$$x^2 - 4x - 5x + 20.$$

Grouping terms, we get:

$$x^2 - 9x + 20 = x(x - 4) - 5(x - 4).$$

$$x^2 - 9x + 20 = (x - 4)(x - 5).$$

The factorised form of the expression $x^2 - 9x + 20$ is $(x - 4)(x - 5)$. □

Example 1.3.12
Factorise the following expression:

$$x^2 + 6x + 9.$$

Solution. Look for two numbers such that the numbers a and b such that $a + b = 6$ and $ab = 9$.
The numbers are 3 and 3.
Then, the expression $x^2 + 6x + 9$ can be written as:

$$x^2 + 3x + 3x + 9.$$

Grouping terms, we get:

$$x^2 + 6x + 9 = x(x + 3) + 3(x + 3).$$

$$x^2 + 6x + 9 = (x + 3)(x + 3).$$

The factorised form of the expression $x^2 + 6x + 9$ is $(x + 3)(x + 3)$. □

Exercises

1. Factorise the following expressions:

(a). $x^2 + 4x + 4$. (e). $x^2 + 3x + 2$. (i). $x^2 - 3x - 10$.
(b). $x^2 + 8x + 15$. (f). $x^2 - 5x + 6$. (j). $x^2 + 7x + 10$.
(c). $x^2 - 7x + 12$. (g). $x^2 + 2x - 15$. (k). $x^2 - 8x - 20$.
(d). $x^2 - 6x + 9$. (h). $x^2 - 4x - 5$. (l). $x^2 + 9x + 20$.

1.3.3.2 Factorisation When The Coefficient Of x^2 Is Not One.

The general form of a quadratic expression is $ax^2 + bx + c$, where a, b and c are constants.

In this section we are going to discuss how to factorise a quadratic expression when the a in $ax^2 + bx + c$ is not equal to 1.

Consider the following identities:

(a) $(3x + 2)(2x + 1)$

$$= (3x \times 2x) + (3x \times 1) + (2 \times 2x) + (2 \times 1)$$

$$= 6x^2 + 3x + 4x + 2$$

$$= 6x^2 + 7x + 2$$

(b) $(3x - 2)(4x - 3)$

$$= (3x \times 4x) - (3x \times 3) - (2 \times 4x) + (2 \times 3)$$

$$= 12x^2 - 9x - 8x + 6$$

$$= 12x^2 - 17x + 6$$

(c) $(2x + 3)(3x - 4)$

$$= (2x \times 3x) - (2x \times 4) + (3 \times 3x) - (3 \times 4)$$

$$= 6x^2 - 8x + 9x - 12$$

$$= 6x^2 + x - 12$$

From the above examples, we can see that the factorisation of a quadratic expression when the coefficient of x^2 is not one is similar to the factorisation of a quadratic expression when the coefficient of x^2 is one.

Factors are multiplied to get the final expresions on the righthand side (RHS) of the equations.

Investigation 1.3.9

Given the quadratic expressions on the right-hand side of the equation, how can you be able factor them?

Consider the expression $6x^2 + 7x + 2$.

The problem can factorised as follows:

(i) Look for two numbers such that:

   (a) Their product is equal to the product of the coefficient of x^2 and the constant term, i.e. $6 \times 2 = 12$.
   In this case 6 is the coefficient of x^2 and 2 is constant term.

   (b) Their sum is 7, where 7 is the coefficient of x.
   The terms are 3 and 4

(ii) Rewrite the term $7x$ as the sum of the two numbers found in step *(i)(b)*.
Thus,

$$6x^2 + 7x + 2 = 6x^2 + 3x + 4x + 2$$

$$= 3x(2x + 1) + 2(2x + 1)$$

$$= (3x + 2)(2x + 1)$$

Example 1.3.13

Factorise the expression $12x^2 - 17x + 6$.

Solution. Multiply the coefficient of x^2 (which is 12) by the constant term (which is 6):

$$12 \times 6 = 72$$

Find two numbers that multiply to give 72 and add up to -17. The two numbers are -8 and -9.

$$-9 \times -8 = 72$$

and

$$-9 + -8 = -17$$

Splt the middle term using these two numbers:

$$12x^2 - 9x - 8x + 6$$

Factor by grouping:

$$3x(4x - 3) - 2(4x - 3)$$

Group the factors:

$$(3x - 2)(4x - 3)$$

□

Example 1.3.14

Factorise the expression $3x^2 - 5x - 2$.

Solution. The expression can be factorised as follows:

1. Look for two numbers such that:
   (a) Their product is equal to the product of the coefficient of x^2 and the constant term, i.e. $3 \times -2 = -6$.
   In this case 3 is the coefficient of x^2 and -2 is constant term.
   (b) Their sum is -5, where -5 is the coefficient of x.
   The terms are -6 and 1
2. Rewrite the term $-5x$ as the sum of the two numbers found in step 1.
   Thus,

$$3x^2 - 5x - 2 = 3x^2 - 6x + x - 2$$

$$= 3x(x - 2) + 1(x - 2)$$

$$= (3x + 1)(x - 2)$$

□

Exercises

1. Factorise the following expressions:

1 (a) $3x^2 + 4x - 7$
(b) $5x^2 - 9x + 4$
(c) $11x^2 - 15x + 10$
(d) $17x^2 - 25x + 16$

2 (a) $2x^2 + 7x + 3$
(b) $13x^2 - 17x + 12$
(c) $6x^2 + 5x - 4$
(d) $4x^2 + 19x - 13$

3 (a) $7x^2 - 13x + 6$
(b) $8x^2 + 3x - 5$
(c) $9x^2 - 14x + 8$
(d) $10x^2 + 11x - 6$

4 (a) $15x^2 - 21x + 14$
(b) $4x^2 - 11x + 6$
(c) $12x^2 + 13x - 11$
(d) $16x^2 + 23x - 15$

1.3.4 Formation of Quadratic Equations by Factorisations.

Activity 1.3.10 ***Work in Groups.***

1. Start by forming a quadratic equation using given roots.
2. Write your equation in factorized form.
3. Swap your equation with another group, solve it and find the roots.

After completing the task, discuss these questions with the class:

- How did you find the quadratic equation from the roots?
- What did you learn about factorization from this exercise?
- Did you notice any patterns when forming quadratic equations?

Finally, each group will share their quadratic equation and the method used with the class.

Any equation of the form $ax^2+bx+c = 0$ where a, b and c are constants and $a \neq 0$ is known as a quadratic equation. In this section, solutions of quadratic equations using the factor method is discussed.

To solve a quadratic equation by factorisation, we aim to rewrite the quadratic equation in the form of:

$$(x + p)(x + q) = 0$$

where p and q are numbers that, when multiplied, give the product c (the constant term), and when added, give the sum b (the coefficient of the middle term).

To solve a quadratic eaquation, first ensure the equations is in the form;

$$ax^2 + bx + c = 0.$$

if not in the form rearrange, so that all terms are on one side of the equation and the equation equals zero.

Factor the quadratic expression:

- Look for two numbers that when multiplied give $a \times c$ (the coefficient of x^2)
- And, when added give b (the coefficient of the middle term).
  - If the quadratic has a leading coefficient $a = 1$, you only need to find two numbers that multiply to c and add to b.
  - If $a \neq 1$, find two numbers that multiply to $a \times c$ and add to b.

Rewrite the middle term using these two numbers and factor by grouping, which you will have a quadratic of the form:

$$(x + p)(x + q) = 0$$

Example 1.3.15

Solve

$$x^2 + 6x + 8 = 0$$

Solution. Factoring the left hand side (L.H.S) of the equation.

$$x^2 + 6x + 8 = 0$$

The factors are $2\,\text{and}4$.

$$x^2 + 2x + 4x + 8 = 0$$

Now create common factors

$$x(x + 2) + 4(x + 2) = 0$$

Grouping them together we have:

$$(x + 2) + (x + 4) = 0$$

□

Example 1.3.16

Form a quadratic expression using the factoring method.

$$6x^2 + 12x$$

Solution. To form a quadratic expression using the factoring method, first set the expression equal to 0 to form a quadratic equation:

$$6x^2 + 12x = 0$$

Remove the common factor out and form a bracket:

$$6x(x + 2) = 0$$

Thus, the factored form of expression $6x^2 + 12x$ is;

$$6x(x + 2) = 0$$

□

Exercises

1. Form a quadratic expression using the following expressions.

(a)
- i $(x + 4)(x + 5)$
- ii $(x + 3)^2$
- iii $4x^2 + 8x$
- iv $3x^2 + 6x$
- v $(p - q)(p - q)$

vi $(ax+b)(2ax-3b)$

(b) i. $(x-8)(x+8)$

ii. $(4-2x)(\frac{1}{2}x+3)$

iii. $(5dx+3d)(2dx-4d)$

iv. $(\frac{1}{2}+x)^2$

v. $(\frac{1}{8}+\frac{1}{x})^2$

1.3.5 Solutions Of Quadratic Equations By Factorisations.

Activity 1.3.11 ***Work in Groups.***

Find the roots of the equation using factorization.

(i). $x^2-5x+6=0$.

(ii). $x^2+7x+10=0$.

After solving the equations, discuss the following:

- What steps did you follow in factorizing the quadratic equation?
- How do the roots relate to the factors of the quadratic equation?
- Did any group use a different method to factorize the equation? Compare approaches.

Finally, each group will share their solved quadratic equations and the methods they used infront of the classroom.

In our previous sections we have discussed the form of a quadratic expression, identities and quadratic equations; how to form quadratic equations and differentiate between an expression and equation

In this section we are going to learn about solving the factored equation, we are going to continue from the point where we form factored equation

Numbers that satisfy an equation (its solutions) are called the ***roots*** of the equation.

Once you have factored the quadratic into the form;

$$(x+p)(x+q)=0$$

Set each factor equal to zero and solve for x:

$$(x+p)=0 \qquad \text{or} \qquad (x+q)=0$$

Solving these will give the two solutions for x.

Example 1.3.17

Solve

$$x^2+5x+6=0 \tag{1.3.1}$$

Solution. Look for two numbers that if multiplied by 6 and when added up gives 5

These numbers are 2 and 3. So we can write the equation as:

$$x^2+2x+3x+6=0$$

Finding a common factor we have:

$$x(x+2)+3(x+2)=0$$

Grouping we have th factors of the equation as:

$$(x+2)(x+3)=0$$

Setting each factor equal to zero (0).

$$x+2=0 \qquad \text{or} \qquad x+3=0$$

Solving for x.

$$x=-2 \qquad \text{or} \qquad x=-3$$

Thus, the solutions to the quadratic equation $x^{+}5x+6=0$ are;

$$x=(-2)\,\text{and}\,x=(-3).$$

□

Example 1.3.18

Solve,

$$6x^2+13x+6=0 \tag{1.3.2}$$

Solution. We need 2 numbers to form factors, which when;

- Multiplied to $6\times 6=36$
- Added up gives 13.

These numbers are 9 and 4. So rewrite the middle part.

$$6x^2+9x+4x+6=0$$

Grouping the factors:

$$(6x^2+9x)+(4x+6)=0$$

Factor each group:

$$3x(2x+3)+2(2x+3)=0$$

$$(2x+3)(3x+2)=0$$

Set each factor eaqual to zero:

$$2x+3=0 \qquad \text{or} \qquad 3x+2=0$$

Solving for x.

$$x=-\frac{3}{2} \qquad \text{or} \qquad x=-\frac{2}{3}$$

Thus, the solutions to $6x^2+13x+6=0$ are:

$$x=-\frac{3}{2}\,\text{and}\,x=-\frac{2}{3}$$

□

Exercises

1. Solve the quadratic equation by factorisation.

(i) $x^2 + 7x + 10 = 0$

(ii) $x^2 - 5x + 6 = 0$

(iii) $x^2 + 3x - 4 = 0$

(iv) $2x^2 + 9x + 7 = 0$

(v) $x^2 - 6x + 8 = 0$

2.

(a) A car's speed is represented by a quadratic equation:

$$4x^2 - 16x + 15 = 0.$$

Find the possible values of x representing time.

(b) The sum of a number and its square is 42. Form a quadratic equation and solve it to find the number.

(c) Solve the quadratic equation:

$$3x^2 - 14x + 8 = 0$$

(d) A garden's area is 56 square meters, and its length is 4 meters more than its width. Form and solve a quadratic equation to find the dimensions of the garden.

1.3.6 Application Of Quadratic Equations To Real Life Situations.

Quadratic equations are a type of mathematical equation that can be used to describe many different ***real-life*** situations. Whether you're throwing a ball in the air, trying to maximize the area of a garden, or calculating profits for a business, quadratic equations can help solve problems that involve relationships with squared terms.

In this section, we will learn how to recognize real-life situations that can be modeled by quadratic equations, how to set them up, and how to solve them step by step.

A quadratic equation is an equation that can be written in the form:

$$ax^2 + bx + c = 0$$

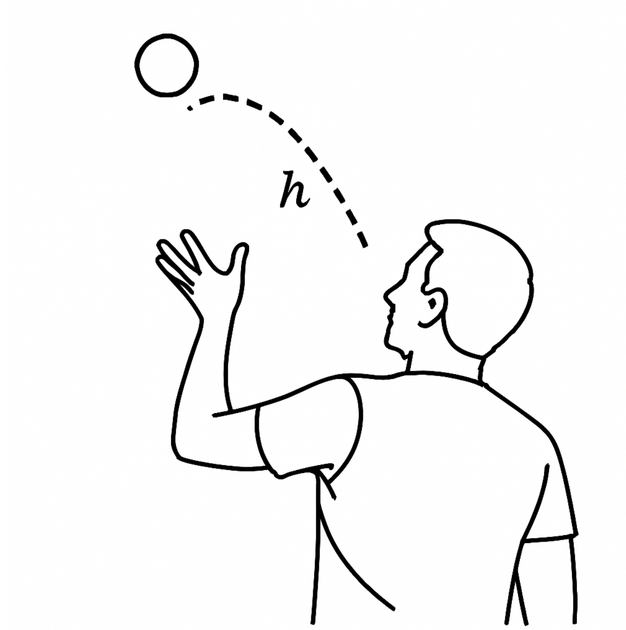
where,

- a, b and c are real numbers or constants,
- x is the unknown variable
- and $a \neq 0$ beacuse if a is 0, it would npt be a quadratic

Quadratic equations show up in many situations, especially when something is changing in a way that involves squares (like height, area, or profit).

Lets take a look at some of the situations ***Quadratic Equations*** is involved in real life situations.

1. $Projectile\, Motion(Throwing\, a\, Ball)$.



One of the most common real-life situations for quadratic equations is **projectile motion**, such as when you throw a ball in the air. The height of the ball over time can be described by a quadratic equation.

Example 1.3.19

A rock is dropped from a height of 50 meters. Its height above the ground at time t is given by

$$h(t) = 5t^2 + 50.$$

Use factorization to determine how long it will take for the rock to reach the ground.

Solution. The height of the rock above the ground is given by:

$$h(t) = -5t^2 + 50$$

To find when the rock reaches the ground, set $h(t) = 0$:

$$-5t^2 + 50 = 0$$

Simplify the equation:

$$-5t^2 = -50$$

$$t^2 = 10$$

Solve for t:

$$t = \pm\sqrt{10}$$

Since time cannot be negative, $t = \sqrt{10}$.

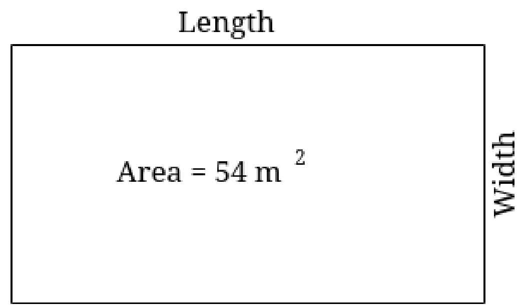
Therefore, the rock will take approximately $\sqrt{10} \approx 3.16$ seconds to reach the ground. □

2. *Maximizing Area* (*Optimization Problem*).

Quadratic equations are also used in optimization problems. These are problems where you want to maximize or minimize something, like the area of a garden.

Example 1.3.20

The length of a rectangle is 3 meters more than its width. If the area of the rectangle is 54 square meters, find the dimensions of the rectangle by factoring the quadratic equation.



Solution. Let the width of the rectangle be x. Then the length is $x + 3$.
The area of the rectangle is given by:

$$x(x + 3) = 54$$

Expand and rearrange into standard quadratic form:

$$x^2 + 3x - 54 = 0$$

Factorize the quadratic equation:

$$(x + 9)(x - 6) = 0$$

Solve for x:

$$x = -9 \,\text{or}\, x = 6$$

Since width cannot be negative, $x = 6$.
The dimensions of the rectangle are:

$$\text{Width} = 6\,\text{meters},\ \text{Length} = 6 + 3 = 9\,\text{meters}.$$

Since the width cannot be negative, the width is 6 meters, and lenght is 6+3 = 9 meters. ☐

3.*Business Applications*(*Profit Function*)

Businesses often use quadratic equations to model their profit. For example, the profit a company makes from selling a product can be modeled by a quadratic equation.

Example 1.3.21

A company's profit $P(x)$ from selling x units of a product is given by the equation:

$$P(x) = -x^2 + 15x - 50$$

Find how many units the company needs to sell to have no profit (i.e., when the profit is zero).

Solution. To find when the company has no profit, set $P(x) = 0$:

$$-x^2 + 20x - 50 = 0$$

Multiply through by -1 to simplify:

$$x^2 - 20x + 50 = 0$$

Factorize the quadratic equation:

$$(x - 10)(x - 5) = 0$$

Solve for x:

$$x = 10 or x = 5$$

Therefore, the company will have no profit when it sells either 5 units or 10 units. □

Exercises

1. A stone is thrown into the air from a height of 4 meters with an initial velocity of 8 meters per second. The height of the stone at time t is given by:

$$h(t) = -5t^2 + 8t + 4$$

Find when the stone reaches the ground.

2. A farmer has 200 meters of fencing. He wants to build a rectangular garden. The length is 50 meters longer than the width. What should the dimensions be to maximize the area?

3. A school's profit function is given by:

$$P(x) = -x^2 + 30x - 100$$

Find the number of units the company must sell to achieve zero profit.

4. A water fountain shoots water into the air, and its height at any time t (in seconds) is given by the equation:

$$h(t) = -4.9t^2 + 15t + 2$$

Find the time it will take for the water to return to the ground (i.e., when $h(t) = 0$).

5. Solve the following quadratic equation:

$$x^2 - 7x + 12 = 0$$

Find the value of x.

6. A company finds that the revenue $R(x)$ it generates from selling x units of a product is given by the quadratic equation:

$$R(x) = 2x^2 + 40x$$

Find the number of units x that the company needs to sell to maximize its revenue.

7. The path of a car is represented by the quadratic equation

$$y = 2x^2 - 8x,$$

where x represents time in seconds and y represents the car's position. Find the time when the car reaches its starting position by factorizing the equation.

Chapter 2

Measurements and Geometry

2.1 Similarity and Enlargement

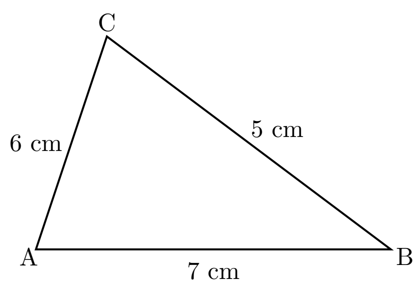
Similarity and enlargement are key concepts in geometry that deal with proportional transformations of shapes.

2.1.1 Similarity

Activity 2.1.1 ***Work in pairs***

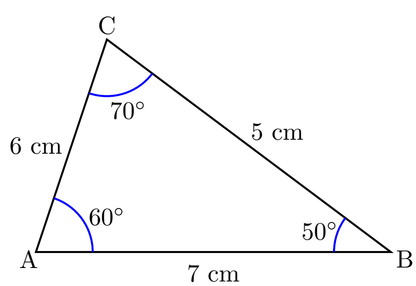
(a) Draw triangle ABC with the following side lengths as shown in the figure below:

$AB = 7$ cm, $AC = 6$cm, and $BC = 5$cm.



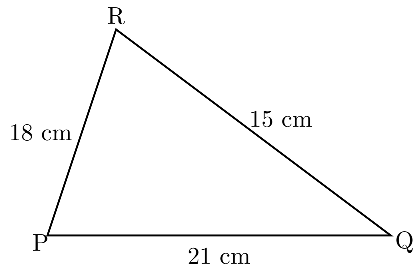
(b) Label the angles in triangle ABC as follows:

$\angle ABC = 50°$, $\angle BAC = 60°$,$\angle BCA = 70°$

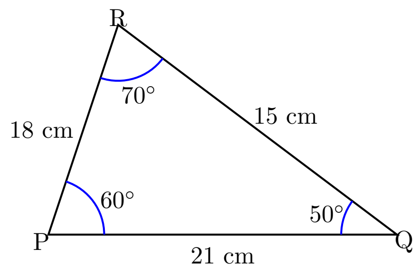


(c) Draw triangle PQR with the following side lengths as shown in the figure below:

$PQ = 21$ cm, $PR = 18$cm, and $QR = 15$cm.



(d) Label the angles in triangle PQR as follows:
$\angle PQR = 50°$, $\angle QRP = 70°$,$\angle QPR = 60°$



(e) Find the ratio of corresponding sides:

- QR to BC (QR/BC).
- PQ to AB (PQ/AB).
- PR to AC (PR/AC).

(f) What do you notice about the ratios of corresponding sides above?

(g) What do you observe between $\angle ABC$ and $\angle PQR$, $\angle BCA$ and $\angle QRP$,$\angle BAC$ and $\angle QPR$

(h) What do you observe about the two triangles based on their corresponding sides and angles?

(i) What is the relationship between triangle ABC and triangle PQR?

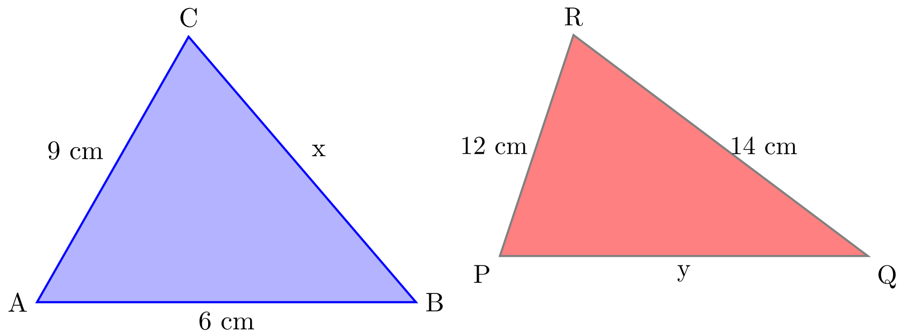
(j) Discuss your findings and share your conclusions with the class.

Key Takeaway

- Two triangles are similar if their corresponding angles are equal and their corresponding sides are in the same ratio.
- Similarity refers to a relationship between two shapes or figures where one can be transformed into the other through scaling (enlargement or reduction), without changing its shape. The two shapes are similar if they have the same shape but may differ in size.

Example 2.1.1

In the figure below, the triangles PQR and ABC are similar. Calculate the lengths marked with letters x and y.



Solution. AB corresponds to PQ, BC corresponds to QR, and AC corresponds to PR.

$$\begin{aligned}
\text{Therefore,}\frac{AB}{PQ} &= \frac{AC}{PR} = \frac{BC}{QR} \\
&= \frac{6\,cm}{y} = \frac{9\,cm}{12\,cm} = \frac{x}{14\,cm} \\
\frac{6\,cm}{y} &= \frac{9\,cm}{12\,cm} \\
y \times 9\,cm &= 6\,cm \times 12\,cm \\
y &= \frac{72\,cm^2}{9\,cm} \\
y &= 8\,cm \\
\text{Therefore PQ } &= 8cm \\
\frac{9\,cm}{12\,cm} &= \frac{x}{14\,cm} \\
x \times 12\,cm &= 9\,cm \times 14\,cm \\
x &= \frac{126\,cm^2}{12\,cm} \\
x &= 10.5\,cm \\
\text{Therefore BC } &= 10.5\,cm
\end{aligned}$$

□

Example 2.1.2

Given that triangles XYZ and PQR in figure 2.1.3 are similar, Find the size of $\angle QPR$, $\angle PQR$ and the length of line PR

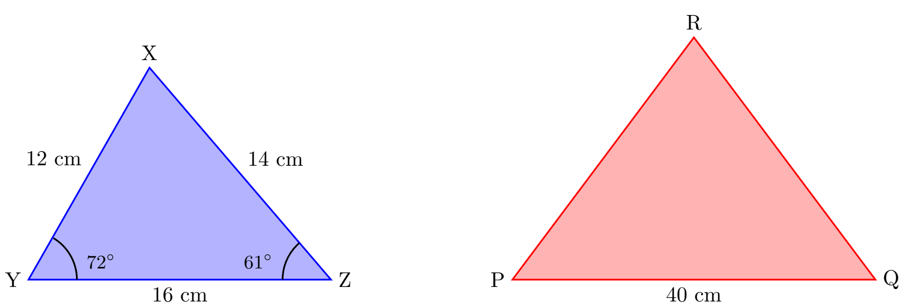


Figure 2.1.3

Solution. Since the two triangles are equal, then their corresponding angles must be equal

$\angle ZYX$ corresponds to $\angle QPR$

Since $\angle ZYX = 72°$, then

$\angle QPR = 72°$

$\angle YZX$ corresponds to $\angle PQR$

Since $\angle YZX = 61°$, then

$\angle PQR = 61°$

Now to find the length of PQ, We use the concept of similarity

$$\begin{aligned} \frac{YZ}{PQ} &= \frac{XY}{PR} \\ \frac{16}{40} &= \frac{12}{PR} \\ PR \times 16 &= 40 \times 12 \\ PR &= \frac{480}{16} \\ PR &= 30\,cm \end{aligned}$$

□

Checkpoint 2.1.4 Using Similar Triangles to Find Unknown Side Lengths. Load the question by clicking the button below.

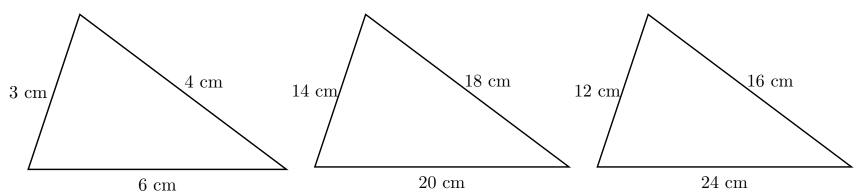
This question contains interactive elements.

Checkpoint 2.1.5 Using Similarity to Determine Unknown Lengths and Scale Factor. Load the question by clicking the button below.

This question contains interactive elements.

Exercises

1. In the triangles below, Determine which triangles are similar by comparing their corresponding sides:



2. Given that triangle ABC is similar to triangle DEF, as shown in the diagram below,

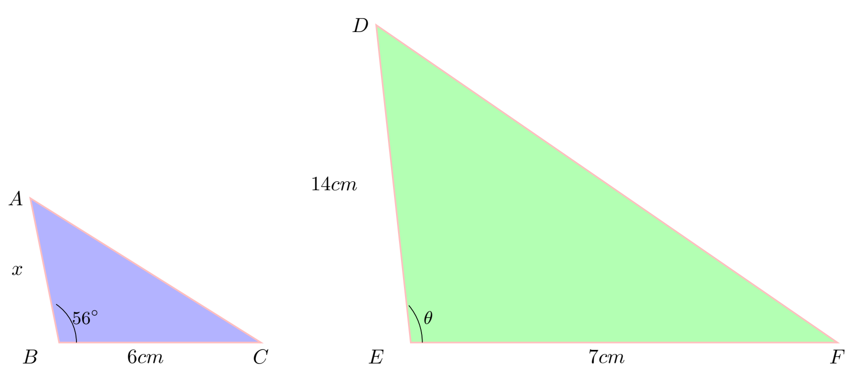


Figure 2.1.6

(a). Determine the measure of angle θ
(b). Calculate the value of x

3. Find the value of x.

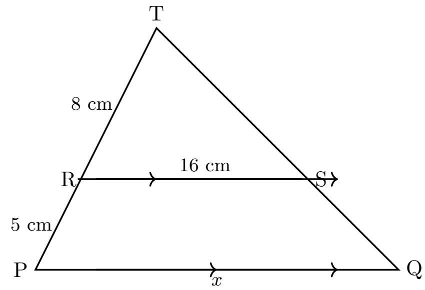


Figure 2.1.7

4. Triangle ABE is similar to triangle ACD, as shown in the figure below, Given that $DC = 24\,cm$, $AE = 6\,cm ED = 12\,cm$, determine the length of BE.

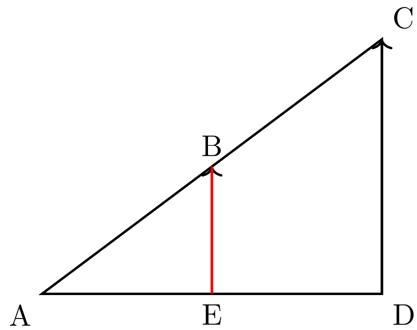
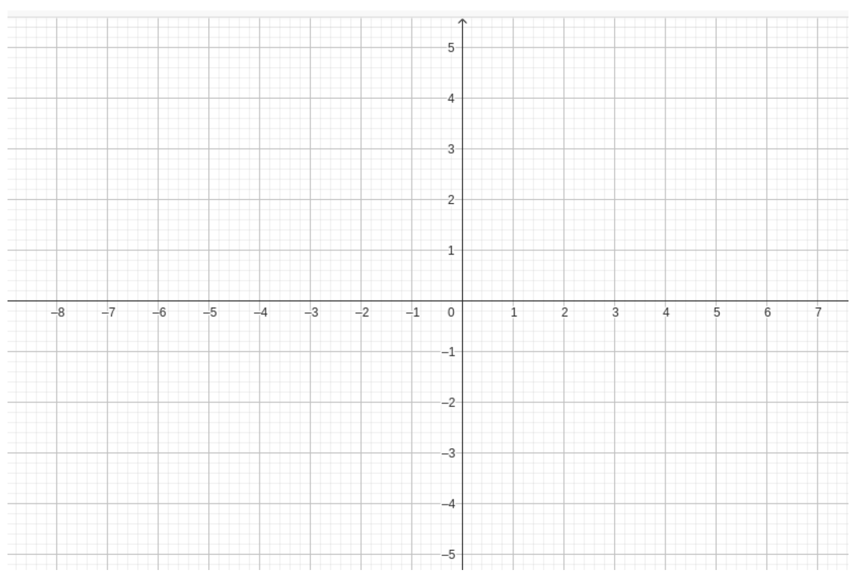


Figure 2.1.8

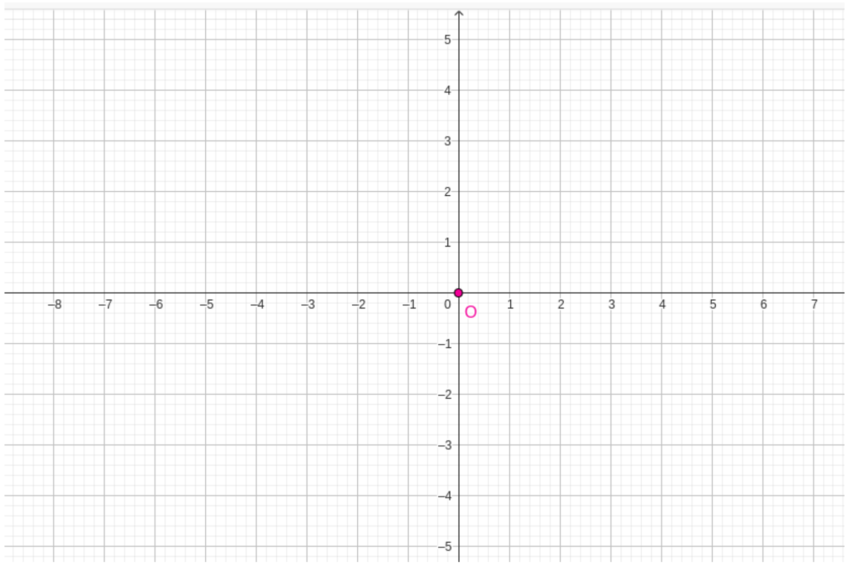
2.1.2 Enlargement

Activity 2.1.2 ***Work in pairs***

(a) Draw and label the x axis and y axis on the graph paper.

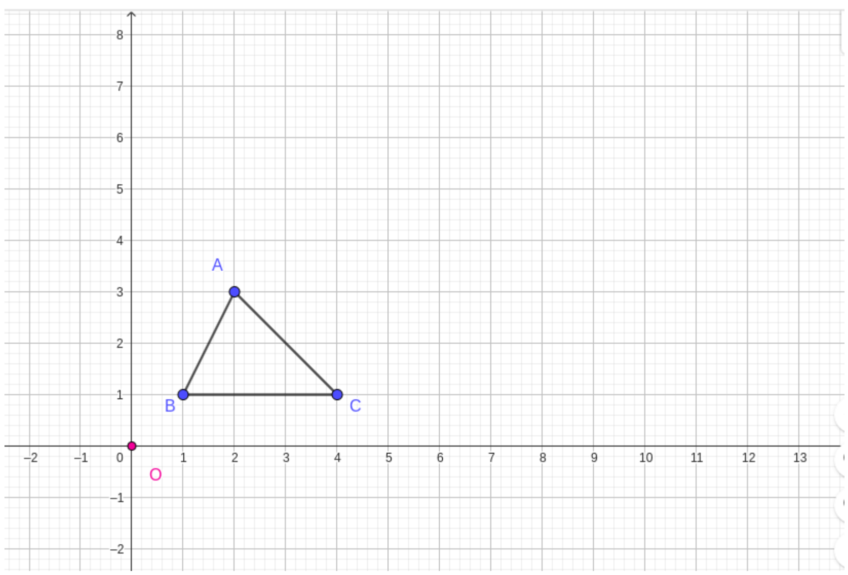


(b) Mark the origin at $(0, 0)$ and label it as **O**.



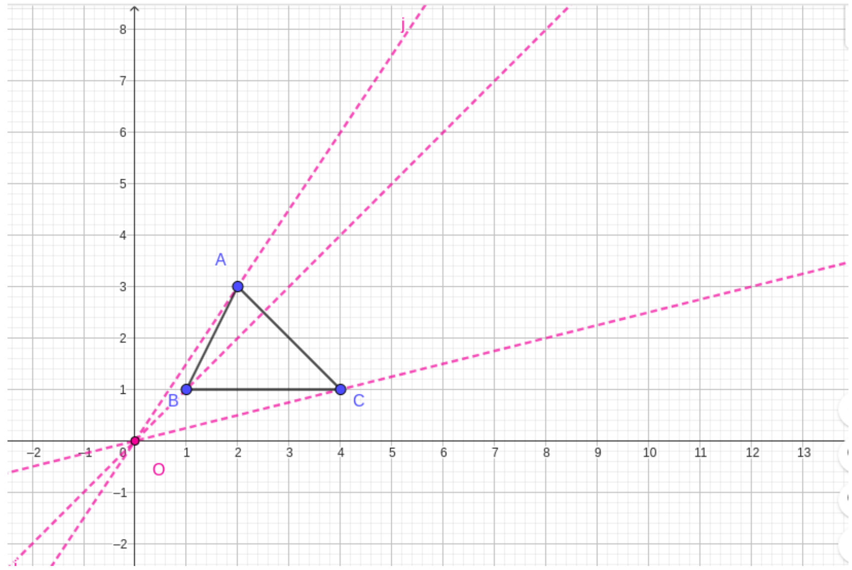
(c) Plot the points $A(2, 3)$, $B(1, 1)$, and $C(4, 1)$.

(d) Connect points A, B, and C with straight lines to form triangle ABC.



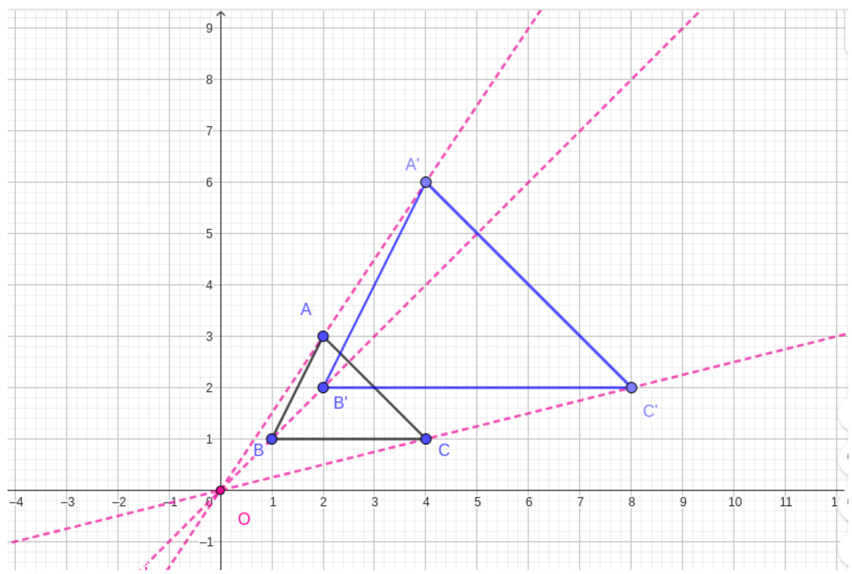
(e) Draw straight lines from O to A, O to B, and O to C.

(f) Measure and record the lengths of OA, OB, and OC.



(g) Extend each line to twice its original length and mark the new points as A', B', and C'.

(h) Connect A', B', and C' to form the enlarged triangle $A'B'C'$.



(i) Compare the two triangles and note any similarities.

(j) Calculate the ratios $\frac{OA'}{OA}$, $\frac{OB'}{OB}$, and $\frac{OC'}{OC}$, what do you notice between the three ratios.

(k) Discuss your findings with the rest of the class.

Key Takeaway

- The process of obtaining triangle $A'B'C'$ from triangle ABC is known as **enlargement.**
- Triangle ABC is said to be object and triangle $A'B'C'$, its image under enlargement.The point O is known as the **centre of enlargement.**
- To determine the scale factor, divide the length of the enlarged image by

the corresponding length of the original object.

$$\frac{OA'}{OA} = \frac{OB'}{OB} = \frac{OC'}{OC}$$

$$\frac{A'B'}{AB} = \frac{A'C'}{AC} = \frac{B'C'}{BC}$$

- Enlargement is a transformation that increases the size of a shape. The shape is enlarged by a scale factor. The scale factor is used to multiply the length of each side of the shape to get the length of the corresponding side of the enlarged shape.
- In an enlargement, the object and its image remain similar. The linear scale factor of the enlargement determines the proportional transformation.
- Lines connecting object points to their corresponding image points intersect at the center of enlargement. This property helps in determining the center of enlargement when both the object and its image are given.

Example 2.1.9

In the figure below, Triangle $P'Q'R'$ is the enlarged image of triangle PQR, with center O

(a) Given that $OP = 6\ cm$ and $PP' = 9\ cm$, determine the linear scale factor of the enlargement.

(b) If $QR = 4\ cm$, find the length of $Q'R'$

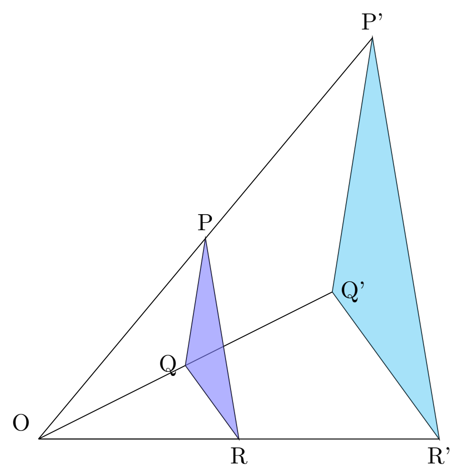


Figure 2.1.10

Solution.

(a). Linear scale factor is:

$$\begin{aligned}\frac{OP'}{OP} &= \frac{(6+9)}{6}\\ &= \frac{15}{6}\\ &= \frac{5}{2}\end{aligned}$$

$$\text{(b).Linear scale factor } = \frac{Q'R'}{QR}$$

$$\text{But QR} = 4\,cm$$
$$\text{Therefore, } \frac{Q'R'}{QR} = \frac{5}{2}$$
$$Q'P' = \frac{(4 \times 5)}{2}$$
$$= \frac{20}{2}$$
$$Q'P' = 10\,cm$$

□

Example 2.1.11

Construct any triangle XYZ and choose a point O outside the triangle. Using O as the center of enlargement and a scale factor of 4, construct the enlarged image of triangle XYZ under the enlargement.

Solution.

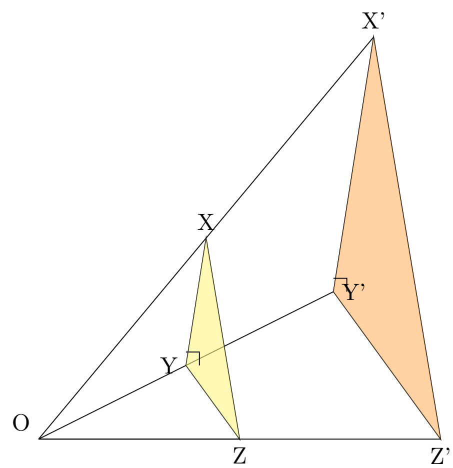


Figure 2.1.12

By measurement;
$OX = 1.8\ cm$, $OY = 3.2\ cm$ and $OZ = 2.7\ cm$.

To determine X', the image of X, We follow these steps:

$$OX = 1.8\,cm$$
$$\frac{OX'}{OX} = \text{scale factor}$$
$$\frac{OX'}{1.8} = 4$$
$$OX' = 4 \times 1.8\,cm$$
$$= 7.2\,cm$$

Extend OX and measure $7.2\ cm$ from O to get X'.

To determine Y', the image of Y, We follow these steps:

$$OY = 3.2\,cm$$
$$\frac{OY'}{OY} = \text{scale factor}$$

$$\frac{OY'}{3.2} = 4$$
$$OX' = 4 \times 3.2\,cm$$
$$= 12.8\,cm$$

Extend OY and measure 12.8 cm from O to get Y'.

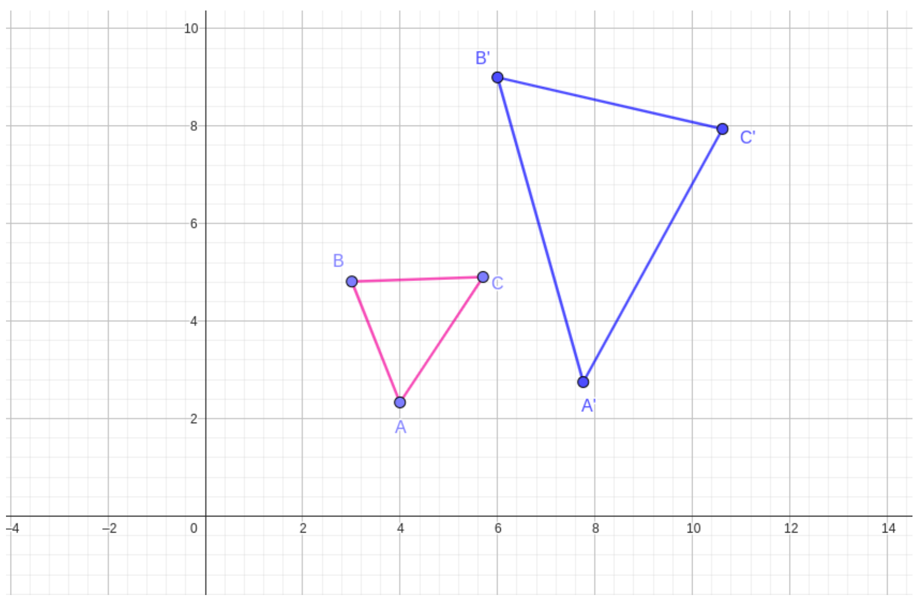
To determine Z', the image of Z, We follow these steps:

$$OZ = 2.7\,cm$$
$$\frac{OZ'}{OZ} = \text{scale factor}$$
$$\frac{OZ'}{2.7} = 4$$
$$OX' = 4 \times 2.7\,cm$$
$$= 10.8\,cm$$

Extend OZ and measure 10.8 cm from O to get Z'. □

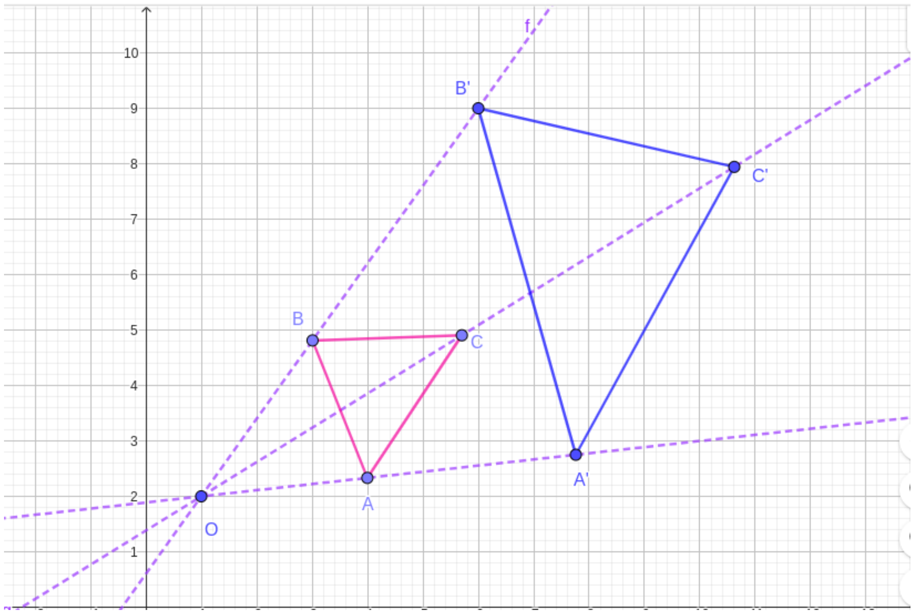
Example 2.1.13

Triangle $A'B'C'$ is the image of triangle ABC under an enlargement.Locate the centre of the enlargement.



Solution. To locate the centre of enlargement we follow the following steps:

Draw lines connecting point A to A', B to B' and C to C' and extend those lines. The intersection point of those lines, will be the center of enlargement.



□

Example 2.1.14

Given that $A\,(6,8)$, $B\,(8,8)$, $C\,(12,8)$, $D\,(14,2)$ and $E\,(10,0)$ are the Vertices of the pentagon, find the vertices of its image after an enlargement with origin as the centre and scale factor of:

(a) 2

(b) $\frac{1}{2}$

(c) -1

Solution. Given the centre of enlargement is $(0,0)$ and the scale factor k,We find the coordinates of the image as follows;

$(x',y') = (kx,ky)$

Applying this to the given co-odinates we get;

(a) for a scale factor of 2

$$\begin{aligned}
A' =&(2\times 6),(2\times 8) = (12,16)\\
B' =&(2\times 8),(2\times 8) = (16,16)\\
C' =&(2\times 12),(2\times 2) = (24,16)\\
D' =&(2\times 14),(2\times 8) = (28,4)\\
E' =&(2\times 10),(2\times 0) = (20,0)
\end{aligned}$$

Vertices of the image $=A'(12,16), B'(16,16), C'(24,16), D'(28,4)andE'(20,0)$

(b) for a scale factor of $\frac{1}{2}$

$$\begin{aligned}
A' =&(\frac{1}{2}\times 6),(\frac{1}{2}\times 8) = (3,4)\\
B' =&(\frac{1}{2}\times 8),(\frac{1}{2}\times 8) = (4,4)\\
C' =&(\frac{1}{2}\times 12),(\frac{1}{2}\times 2) = (6,4)\\
D' =&(\frac{1}{2}\times 14),(\frac{1}{2}\times 8) = (7,1)
\end{aligned}$$

$$E' = (\frac{1}{2} \times 10), (\frac{1}{2} \times 0) = (5, 0)$$

vertices of the image $= A'(3,4), B'(4,4), C'(6,4), D'(7,1)$ and $E'(5, O)$

(c) for a scale factor of -1

$$\begin{aligned} A' &= (-1 \times 6), (-1 \times 8) = (-6, -8) \\ B' &= (-1 \times 8), (-1 \times 8) = (-8, -8) \\ C' &= (-1 \times 12), (-1 \times 2) = (-12, -8) \\ D' &= (-1 \times 14), (-1 \times 8) = (-14, -2) \\ E' &= (-1 \times 10), (-1 \times 0) = (-10, 0) \end{aligned}$$

vertices of the image $= A'(-6,-8), B'(-8,-8), C'(-12,-8), D'(-14,-2)$ and $E'(-10,0)$

□

Negative scale factor

In the provided diagram, rectangle $ABCD$ has been enlarged to form rectangle $A'B'C'D'$, with point O as the center of the enlargement.

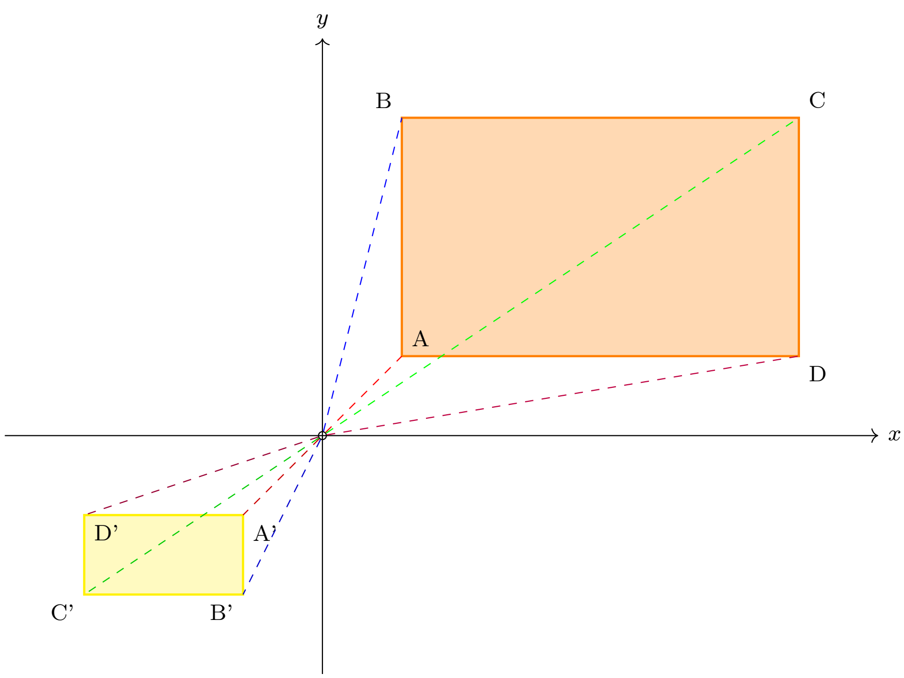


Figure 2.1.15

Note:

If an enlargement has a negative scale factor, the image is formed on the opposite side of the center and is inverted (Upside down).

The rectangle $ABCD$ has been enlarged by a scale factor of $-\frac{1}{2}$.

Example 2.1.16

Enlarge the triangle ABC by scale factor of -1 about the point O.

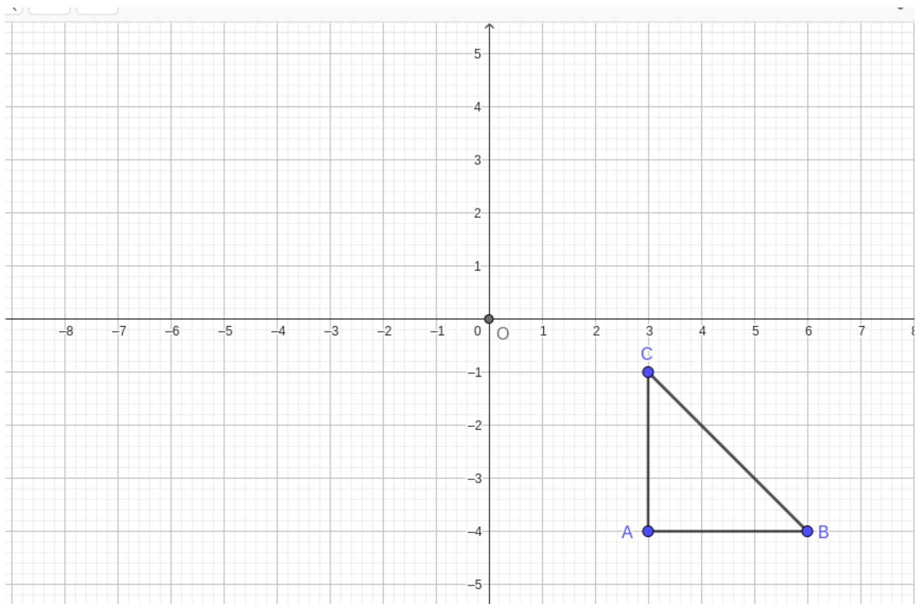
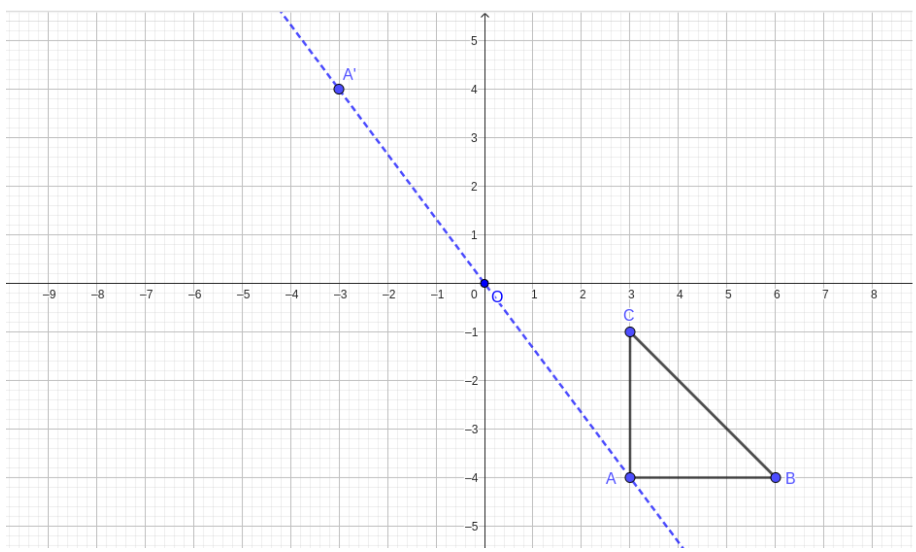


Figure 2.1.17

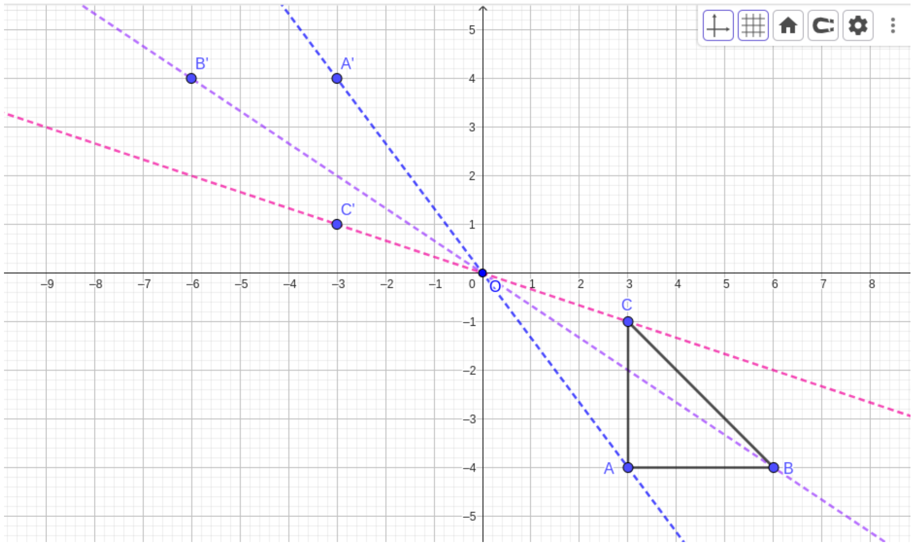
Solution. The centre of enlargement is O, the origin.

Draw a line from point A through O and extend the line upwards through the centre of enlargement.

Measure the distance from point O to point A. Since the scale factor is -1, and the distance from $OA = 5$, then $OA' = -1 \times 5 = -5$



Similarly draw the lines from point B through O and C through O and extend the line upwards through the centre of enlargement. Measure the distance from point O to point B' and O to point C' and multipy by the scale factor -1 to get the new distance from O to point B and O to point C



Join up the points to make the new triangle $A'B'C'$

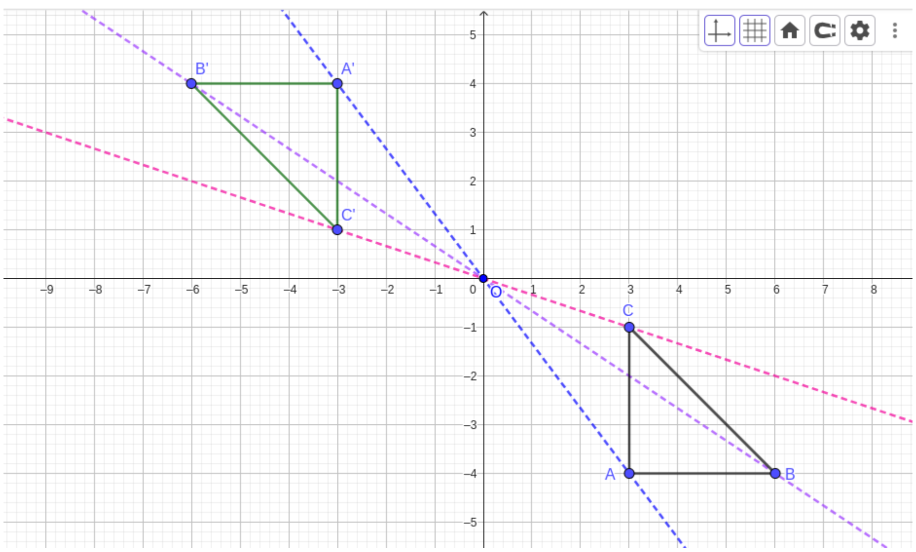


Figure 2.1.18

□

Checkpoint 2.1.19 Enlargements (25-4). Load the question by clicking the button below.

This question contains interactive elements.

Checkpoint 2.1.20 Circumference of a Circle Under Enlargement. Load the question by clicking the button below.The radius of a circle is 22.4 ***cm***. The circle is enlarged by a scale factor of 0.25. Find the circumference of the image of the circle after the enlargement.

_____ cm

Answer. 35.2

Solution. Worked solution The original radius is $r = 22.4$ cm.

After enlargement by a scale factor of 0.25, the new radius is
$r' = 0.25 \times 22.4 = 5.6$ cm.

The circumference of a circle is given by $C = 2\pi r$.

So the circumference of the enlarged circle is

$C = 2 \times \frac{22}{7} \times 5.6 = 35.2$ cm.
Equivalently, since circumference scales linearly,
$C = 0.25 \times (2 \times \frac{22}{7} \times 22.4) = 35.2$ cm.

Checkpoint 2.1.21 Enlargement of a Triangle with Centre at the Origin. Load the question by clicking the button below.The vertices of triangle $\triangle ABC$ are$A\,(-2, 9)$, $B\,(8, 7)$ and$C\,(5, 4)$. The triangle is enlarged with centre $(0, 0)$ and scale factor $\frac{1}{2}$. Find the coordinates of the image of $\triangle ABC$.

Write your answer in form of ***[x,y]***

A' = _____
B' = _____
C' = _____

Answer 1. $\left[-1, \frac{9}{2}\right]$

Answer 2. $\left[4, \frac{7}{2}\right]$

Answer 3. $\left[\frac{5}{2}, 2\right]$

Solution. Worked solution Given the centre of enlargement is $(0, 0)$ and the scale factor k, We find the coordinates of the image as follows;
$(x', y') = (kx, ky)$
Applying this to the given co-odinates we get;
$A' = (\frac{1}{2} \times -2), (\frac{1}{2} \times 9) = (-1, \frac{9}{2})$
$B' = (\frac{1}{2} \times 8), (\frac{1}{2} \times 7) = (4, \frac{7}{2})$
$C' = (\frac{1}{2} \times 5), (\frac{1}{2} \times 4) = (\frac{5}{2}, 2)$
Verticies of the image = A' $(-1,\ \frac{9}{2})$, B' = $(4,\ \frac{7}{2})$ and C' = $(\frac{5}{2},\ 2)$.

Checkpoint 2.1.22 Enlargement and Similarity Parsons Proof. Load the question by clicking the button below.

This question contains interactive elements.

Checkpoint 2.1.23 This question contains interactive elements.

Exercises

1. A triangle with the verticies $X\,(4, 0)$, $Y\,(6, 3)$ and $Z\,(5, 4)$ is enlarged.If the centre of enlargment is $(1, 1)$, find the co-ordinates of the image of the triangle when the scale factor is:

(a) -2 (b) $\frac{1}{2}$

2. Points $A\,(2, 6)$, $B\,(4, 6)$, and $C\,(4, 2)$ are the vertices of a triangle. Taking point $(0, 2)$ as the centre of enlargement, find the coordinates of its image when the scale factor is -1.

3. Points $P\,(1, 4)$, $Q\,(3, 4)$ and $R\,(3, 1)$ are verticies of a triangle. Taking the origin as the centre of enlargement, find the image when the scale factor is;

(a) $-\frac{1}{4}$ (b) -3 (c) 2

4. A square measures $5\,cm$ by $9\,cm$. Find the corresponding measurements of the image of the square after an enlargement with scale factor of -2.

5. A photograph is enlarged so that its width increases from $10\,cm$ to $25\,cm$. If the original height is $15\,cm$, find the new height.

6. A map has a scale of $1 : 50,000$. If the distance between two cities on the map is $8\,cm$, find the actual distance.

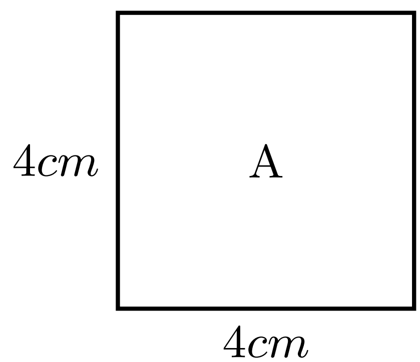
2.1.3 Scale Factors

- Scale factor is a fundamental concept in mathematics, especially in geometry, where it is used to describe the proportional relationship between similar figures.
- It is defined as the ratio of corresponding side lengths in two similar shapes.
- Understanding scale factor helps students grasp how objects are enlarged or reduced while maintaining their shape and proportions.

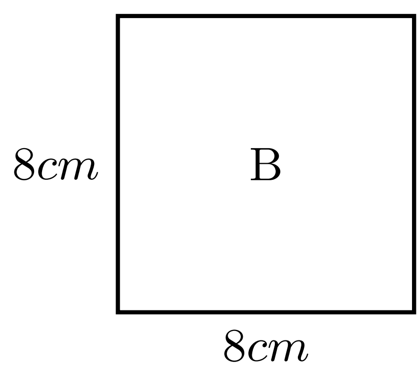
2.1.3.1 Area Scale Factor

Activity 2.1.3 ***Work in pairs***

(a) Draw a square with a side length of 4 cm and label it as Square A as shown below.



(b) Draw another square with a side length of 8 cm and label it as Square B as shown below.



(c) Calculate the area of Square A and Square B.

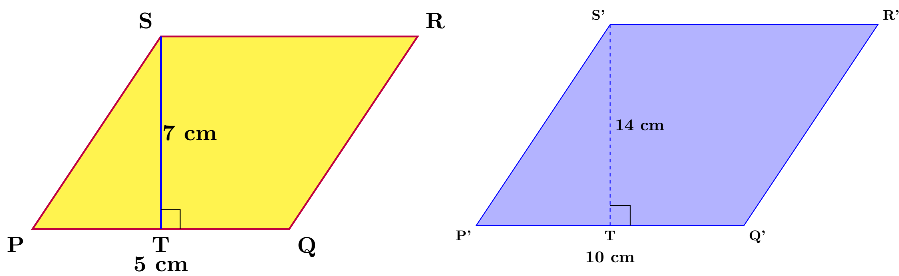
(d) Find the ratio of the areas by dividing the area of Square B by the area of Square A.

(e) Take one side length of Square B, divide it by one side length of Square A, and then square the result.

(f) Compare your answers from steps (**d**) and (**e**). What do you notice?

(g) Discuss and share your findings with the rest of the class.

Key Takeaway
Area scale factor is the ratio of the area of the image to area of the object.
Area scale factor is the square of linear scale factor.
In the Figures below, the parallelogram $P'Q'R'S'$ represents the enlarged image of paralellogram $PQRS$, transformed by a scale factor of 2.



$$\begin{aligned}
\text{Area of a parallelogram } =& base \times height \\
\text{Area of PQRS } =& 5\,cm \times 7\,cm \\
=& 35\,cm^2 \\
\text{Area of P'Q'R'S' } =& 10\,cm \times 14\,cm \\
=& 140\,cm^2 \\
\text{Area scale factor } =& \frac{\text{Area of the image}}{\text{area of the object}} \\
\frac{\text{Area of P'Q'R'S'}}{\text{Area of PQRS}} =& \frac{140\,cm^2}{35\,cm^2} \\
=& 4
\end{aligned}$$

Example 2.1.24
A square whose area is 28 cm^2 is given an enlargement with a linear scale factor of 4. Find the area of the image.

Solution.

$$\begin{aligned}
\text{Area scale factor } =& \frac{\text{Area of image}}{\text{area of object}} \\
\text{Linear scale factor (L.S.F) } =& 4 \\
\text{Area scale factor(A.S.F) } =& (L.S.F)^2 \\
=& 4^2 = 16 \\
\frac{\text{Area of image}}{28} =& 16 \\
\text{Area of image } =& 16 \times 28 \\
=& 448\,cm^2
\end{aligned}$$

□

Example 2.1.25
Given that the ratio of the area of two circles is $\frac{25}{64}$.
(a). Find the ratio of their radii
(b). If the smaller one has a radius of 15 cm, find the radius of the larger one.

Solution.

$$\text{(a)Area scale factor } = \frac{25}{64}$$

$$\text{Linear scale factor} = \sqrt{\frac{25}{64}} = \frac{5}{8}$$

Therefore the ratio of two radii $= 5 : 8$

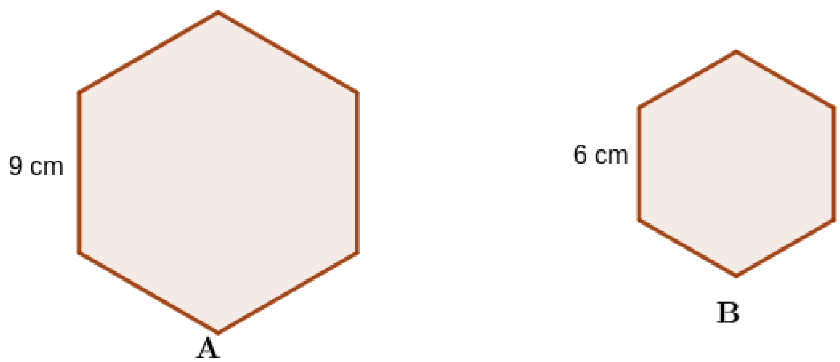
(b).If the radius of the smaller circle is $15\,cm$, then

$$\frac{\text{Radius of smaller circle}}{\text{Radius of larger circle}} = \frac{5}{8}$$
$$\frac{15}{\text{radius of larger circle}} = \frac{5}{8}$$
$$\text{radius of larger circle} = \frac{(8 \times 15)}{5} = 24\,cm$$

□

Example 2.1.26

Given that the following two hexagons below are similar, and the area of the first hexagon A is $450\,cm^2$, calculate the area of the second hexagon B.



Solution.

$$\text{Linear scale factor} = \frac{9}{6}$$
$$\text{Area scale factor} = (\frac{9}{6})^2 = \frac{81}{36}$$
$$\frac{\text{Area of hexagon A}}{\text{Area of hexagon B}} = \frac{81}{36}$$
$$\frac{450}{\text{Area of hexagon B}} = \frac{81}{36}$$
$$\text{Area of hexagon B} = \frac{36 \times 450}{81} = 200 cm^2$$

□

Checkpoint 2.1.27 Finding the Area of a Similar Polygon. Load the question by clicking the button below.The corresponding sides of two similar regular polygons are 5 cm and 6 cm respectively.The area of the smaller polygon is $80\,\text{cm}^2$. What is the area of the larger polygon?

Note: Write your answer correct to ***2*** decimal places.

_____ cm^2

Answer. 115.2

Solution. Worked solution Since the two polygons are similar, all corresponding lengths are in the same ratio.Here, the ratio of corresponding sides is:

$\frac{6}{5}$.

For similar figures, areas scale with the square of the ratio of corresponding lengths.Therefore, the ratio of the areas is:

$\left(\frac{6}{5}\right)^2$.

If the area of the smaller polygon is 80, then the area of the larger hexagon is

$80 \times \left(\frac{6}{5}\right)^2 . = 115.2\,\text{cm}^2$

Checkpoint 2.1.28 Finding the Area of a Smaller Cone. Load the question by clicking the button below.The ratio of the area of two similar cones is $\frac{16}{484}$.

(a). Find the area of the smaller cone if the area of the bigger cone is 880 m^2. ***Note:*** Write your answer correct to ***2***decimal places.

_____ m^2

(b). If the slanting height of the smaller cone is 17 m find the slanting height of the larger cone.***Note:*** Write your answer correct to ***2*** decimal places.

_____ m

Answer 1. 29.09

Answer 2. 93.5

Solution. Worked solution

$$\begin{aligned}\text{(a).Area of the smaller cone} &= \frac{16 \times 880}{484}\\ &= 29.0909090909\\ &= 29.09\end{aligned}$$

$$\begin{aligned}\text{(b).Slant height of the larger cone} &= \frac{484 \times 17}{16}\\ &= 93.5\end{aligned}$$

Checkpoint 2.1.29 Relationship Between Side Lengths and Areas. Load the question by clicking the button below.The corresponding sides of two similar regular hexagons are 7 and 13 respectively.

(a) Find the ratio of their areas. _____

(b). If the area of the smaller hexagon is 14, calculate the area of the larger hexagon. ***Note:*** Write your answer correct to 2 decimal places _____

.

Answer 1. $\frac{169}{49}$

Answer 2. 48.29

Solution. Worked Solution Determine the linear scale factor

The side lengths of the similar regular hexagons are:

Small hexagon: 7

Large hexagon: 13

The linear scale factor from the smaller to the larger hexagon is

$k = \frac{13}{7}$.

Ratio of their areas

For any pair of similar shapes, the ratio of their areas is the square of the ratio of their corresponding side lengths. Thus,

Area ratio $= k^2 = \left(\frac{13}{7}\right)^2 = \frac{169}{49}$.

Area of the larger hexagon

We are told that the area of the smaller hexagon = 14.

To find the area of the larger hexagon, multiply by the area ratio:

$A_{\text{large}} = A_{\text{small}} \cdot k^2 = 14 \cdot \frac{169}{49} = \frac{338}{7}$.

Exercises

1. The corresponding sides of two similar regular hexagons are $4\,cm$ and $9\,cm$ respectively.
(a).Find the ratio of their areas.
(b).Calculate the area of the larger hexagon if the area of the smaller hexagon is $64\,cm^2$.

2. The ratio of the area of two similar cones is $\frac{9}{36}$.
(a).Find the area of the smaller cone if the area of the bigger cone is $320\,m^2$
(b). Find the ratio of their base radii.
(c). If the slanting height of the smaller cone is $7\,m$, find the slanting height of the larger cone.

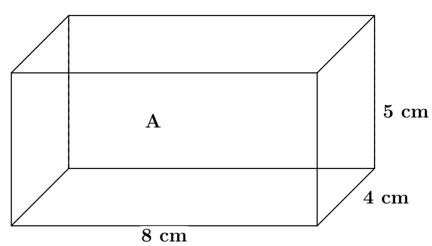
3. The length of a parallelogram is $15\,cm$ and its area is $240\,cm^2$. Calculate the length of a similar parallelogram whose area is $375\,cm^2$.

4. The area of a circle is $49\,m^2$. A second circle has a radius that is 4 times the radius of the first circle. What is the area of the second circle?

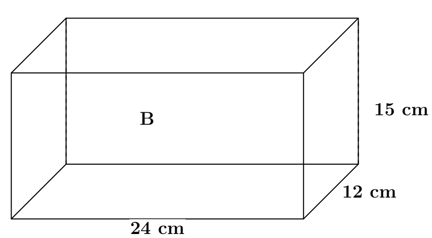
2.1.3.2 Volume Scale Factor

Activity 2.1.4 ***Work in pairs***

(a) Draw a cuboid with dimensions 8 cm (length), 5 cm (width), and 4 cm (height), and label it as Cuboid A as shown below.



(b) Draw another cuboid with dimensions 24 cm (length), 15 cm (width), and 12 cm (height), and label it as Cuboid B as shown below.



(c) Calculate the volume of each cuboid by multiplying length × width × height.

(d) Find the volume ratio by dividing the volume of Cuboid B by the volume of Cuboid A.

(e) Find the ratio of the length of Cuboid B to the length of Cuboid A, then raise the result to the power of three

(f) Calculate the ratio by dividing the width of Cuboid B by the width of Cuboid A, then cube the resulting value

(g) Determine the ratio by dividing the height of Cuboid B by the height of Cuboid A, then cube the resulting value.

(h) Compare the result obtained from step **d** with the values calculated in steps **e**, **f**, and **g**. Note any patterns or relationships you observe among these results.

(i) Discuss and share the results with the rest of the learners in the class.

Key Takeaway

A volume scale factor is the cube of the linear scale factor, representing the ratio by which the volume of a scaled object changes compared to the original object.

Example 2.1.30

The corresponding heights of two similar cylinders are $4\,m$ and $5\,m$.
(a). Find the ratio of their corresponding volumes.
(b).If the smaller cylinder has a volume of $1536\,m^3$, find the volume of the larger cylinder.

Solution.

$$\begin{aligned}
\text{(a).The ratio of the heights} =& \frac{4}{5}\\
\text{Linear scale factor} =& \frac{4}{5}\\
\text{Volume scale factor} =& (L.S.F)^3\\
=& \left(\frac{4}{5}\right)^3\\
=& \frac{64}{125}\\
\text{Therefore ratio of the volumes } =& 64:125\\
\text{(b). Volume of larger cylinder;}&\\
\frac{\text{Volume of smaller cylinder}}{\text{volume of larger cylinder}} =& \text{volume scale factor}\\
\frac{1536}{\text{Volume of larger cylinder}} =& \frac{64}{125}\\
\text{Volume of larger cylinder} =& \frac{(125\times 1546)}{64}\\
=& 3000\,m^3
\end{aligned}$$

□

Example 2.1.31

The capacity of two similar containers are $288\,cm^3$ and $4500\,cm^3$.Find the ratio of their:
(a). Heights
(b). If the area of the smaller container is $140\,cm^2$, find the area of the

larger container.

Solution.

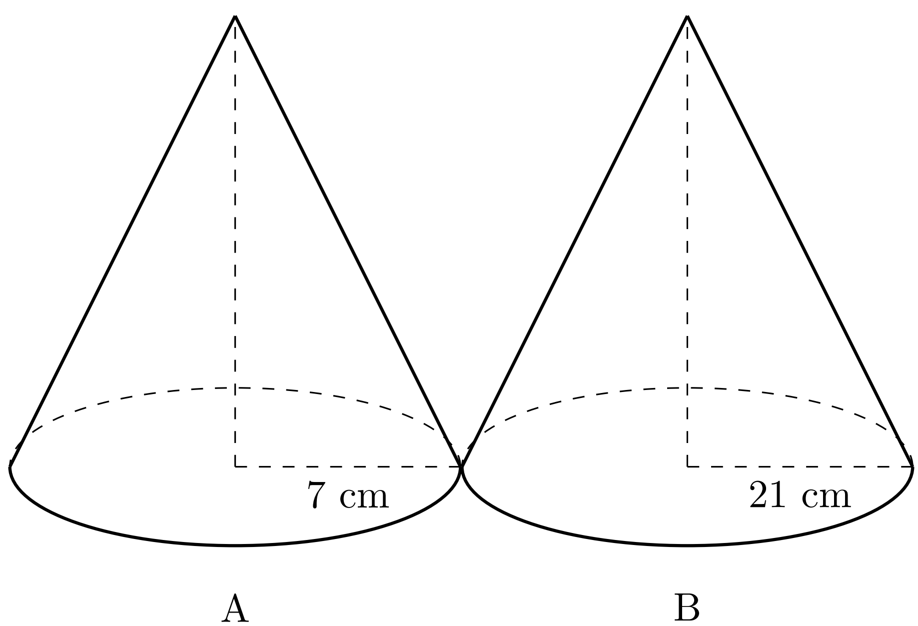
$$\text{Volume scale factor} = \frac{288\,cm^3}{4500\,cm^3} = \frac{8}{125}$$
$$\text{Lenear scale factor} = \sqrt[3]{\frac{8}{125}}.$$
$$= \frac{2}{5}$$
$$\text{Therefore the ratio of the heights} = 2:5$$

$$\text{Area scale factor} = \left(\frac{2}{5}\right)^2$$
$$= \frac{4}{25}$$
$$\text{Area scale factor} = \frac{\text{Area of the smaller container}}{\text{Area of the larger container}}$$
$$\frac{4}{25} = \frac{140}{\text{Area of larger container}}$$
$$\text{Area of larger container} = \frac{(25 \times 140)}{4}$$
$$= 875$$
$$\text{The area of larger container} = 875\,cm^2$$

□

Example 2.1.32

Given that cone A and cone B are similar cones, and the volume of cone A is $150cm^3$, calculate the volume of cone B.



Solution.

$$\begin{aligned}\text{Linear scale factor} =& \frac{7}{21} = \frac{1}{3}\\ \text{Volume scale factor} =& (\frac{1}{3})^3 = \frac{1}{27}\\ \frac{\text{Volume of cone A}}{\text{Volume of cone B}} =& \frac{1}{27}\\ \frac{150}{\text{Volume of cone B}} =& \frac{1}{27}\\ \text{Volume of cone B} =& 27 \times 150\\ =& 4050 cm^3\end{aligned}$$

□

Checkpoint 2.1.33 Similar Pyramids (28-2). Load the question by clicking the button below.

This question contains interactive elements.

Checkpoint 2.1.34 Finding the Capacity of a Larger Container. Load the question by clicking the button below.Two similar containers have heights of 3 cm and 6 cm respectively. If the smaller container holds 295 ml, what is the capacity of the larger container? ***Note:*** Write your answer correct to ***2*** decimal places.

_____ ml

Answer. 2360

Solution. Worked solution Given the linear scale factor (L.S.F)= $\frac{6}{3}$

Volume scale factor (V.S.F) = $(L.S.F)^3 = (\frac{6}{3})^3 = 8$

Given the capacity of the small container as 295 ml, then the capacity of larger container is given by;

$$\frac{216}{27} = \frac{x}{295} \Rightarrow x = \frac{295 \times 216}{27} = 2360 = 2360 \text{ ml}$$

Checkpoint 2.1.35 Finding the Height of a Larger Can. Load the question by clicking the button below.Two similar cans have volumes of 125 m^3 and 1728 m^3respectively. If the smaller can has a height of 15 m, what is the height of the larger can?

_____ m

Answer. 36

Solution. Worked solution Since the two cans are ***similar solids***, the ratio of their corresponding linear dimensions is equal to the ***cube root of the ratio of their volumes***.

$\frac{H_{\text{large}}}{H_{\text{small}}} = \sqrt[3]{\frac{V_{\text{large}}}{V_{\text{small}}}}$

Substituting the given values: $\frac{H_{\text{large}}}{15} = \sqrt[3]{\frac{1728}{125}} = \frac{12}{5}$

Hence, the height of the larger can is: $H_{\text{large}} = \frac{12}{5} \times 15 = 36.0\,\text{m}$

Rounding to two decimal places, the final answer is: $\boxed{36\,\text{m}}$

Checkpoint 2.1.36 Ratio of Volumes of Similar Solids. Load the question by clicking the button below.The corresponding sides of two ***similar*** blocks of wood measure 9 cm and 10 cm. Find the ratio of their volumes.

_____

Answer. $\frac{1000}{729}$

Solution. Worked solution Since the blocks are ***similar***, all corresponding lengths scale by the same factor.

The linear scale factor is:

$$\frac{10}{9}$$

For similar solids, volumes scale with the ***cube*** of the linear scale factor:

$$\left(\frac{10}{9}\right)^3 = \frac{1000}{729}.$$

Therefore, the ratio of the volumes of the smaller block to the larger block is:

$$1 : \frac{1000}{729} = \frac{1000}{729}.$$

Checkpoint 2.1.37 Similarity and Enlargement - Cylinders. Load the question by clicking the button below.

This question contains interactive elements.

Exercises

1. Two similar containers have heights of $6\,cm$ and $9\,cm$, respectively. If the smaller container holds $400\,ml$, what is the capacity of the larger container?

2. Two similar cans have volumes of $192\,cm^3$ and $648\,cm^3$ respectively. If the smaller can has a height of $14\,cm$, what is the height of the larger can?

3. The ratio of the lengths of the corresponding sides of two similar rectangular tanks is $3 : 5$.The volume of the smaller tank is $8\,cm^3$.Calculate the volume of the larger tank.

4. A small cube has a length of $3\,cm$. A larger cube is created by scaling the small cube, such that each side of the larger cube is 6 times the length of the corresponding side of the small cube.

(a).What is the volume of the small cube?

(b).What is the volume of the larger cube?

(c). By what factor has the volume increased when the small cube is scaled to the larger cube?

2.1.3.3 Further Exercise

1. An architect is creating a scale model of a building. The actual height of the building is 120 meters, and the height of the model is 0.6 meters.

(a) What is the scale factor of the model?

(b) If the width of the actual building is 50 meters, what is the width of the model?

2. A map scale is given as $1 : 25,000$, meaning $1\,cm$ on the map represents $25,000\,cm$ in real life.

(a) A river on the map measures $8\,cm$ in length. What is the actual length of the river in kilometers?

(b) If a road on the map measures $12.5\,cm$, how long is the actual road in meters?

3. A photograph has a size of $5\,cm$ by $7\,cm$. It needs to be enlarged so that the width becomes $20\,cm$. The height will also increase proportionally. What is the new height of the photograph after the enlargement?

4. A pole of height 2.4 meters casts a shadow of length 1.6 meters. A tree casts a shadow of length 12 meters.

(a) Using the concept of similar triangles, find the height of the tree.

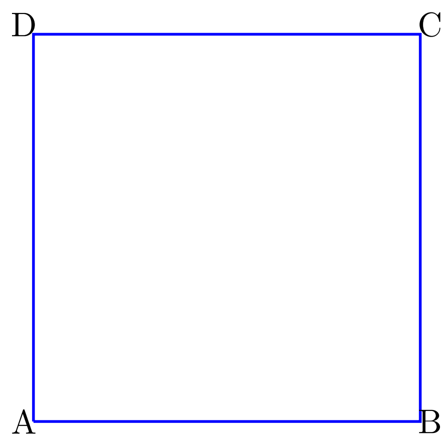
(b) If the tree's shadow increases to 15 meters, what would be the new height of the tree, assuming the proportion remains the same?

2.2 Reflection and Congruence

2.2.1 Identifying Lines of Symmetry Given an Object

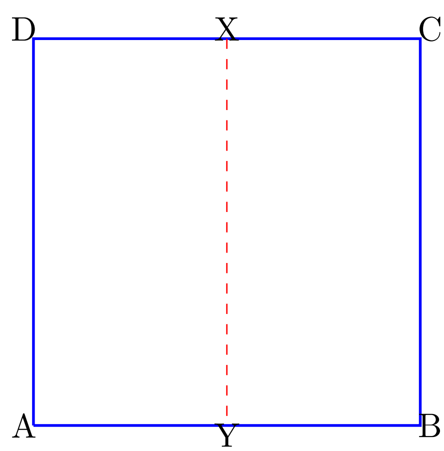
Work in groups

Activity 2.2.1 From a piece of paper, cut out a square shape and label it as shown below.



Fold the square in half from left to right such that the corner A aligns with corner B and corner D aligns with corner C. This will create a rectangle.

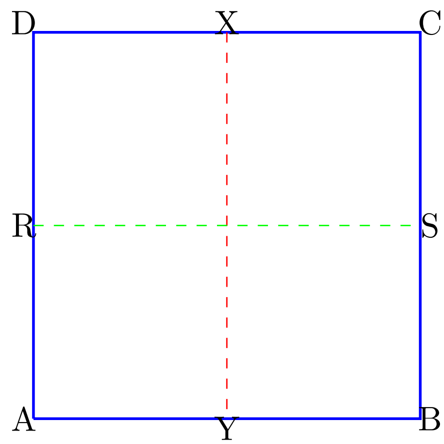
Unfold the paper and make a dotted line across the fold line and label it as XY as shown.



Notice that the left side of the line XY and the right side are exacly the same/ identical.

Now, fold the square in half from the top to the bottom such that the corner D aligns with corner A and corner C aligns with corner B. This will create another rectangle.

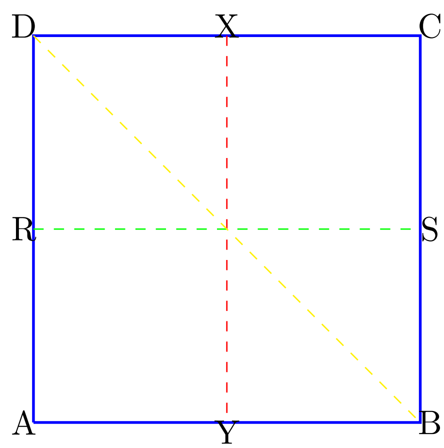
Unfold the paper and make a dotted line across the second fold line and label it as RS as shown.



Again, you notice that the upper side of the line RS and the lower side are exacly the same/ identical.

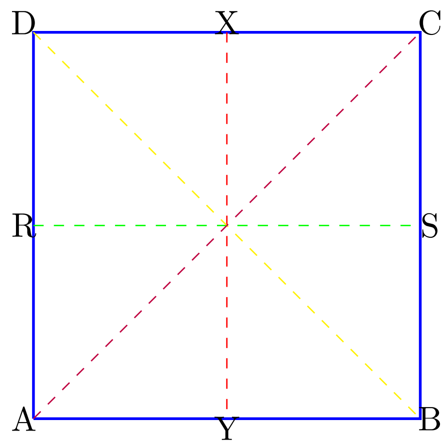
Next, fold the paper in half from the bottom left corner A to the top right corner C. This creates a triangle.

Unfold the paper, you will notice that a fold line appears along BD. Trace a dotted line along the fold line as showm.



Finally, fold the paper in half from the top left corner D to the bottom right corner B. This creates another triangle.

Now, unfold the paper, you will notice that a fold line appears along AC. Trace a dotted line along the fold line as shown.



Key Takeaway

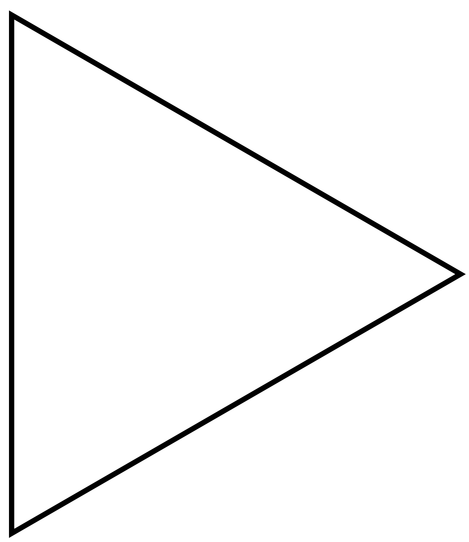
There are four dotted lines XY, RS, AC and BD. Both sides of each line are exactly the same.

Therefore **symmetry** is when an object/shape looks exactly similar or identical on one side and the other side when the object is folded/flipped, rotated or reflected.

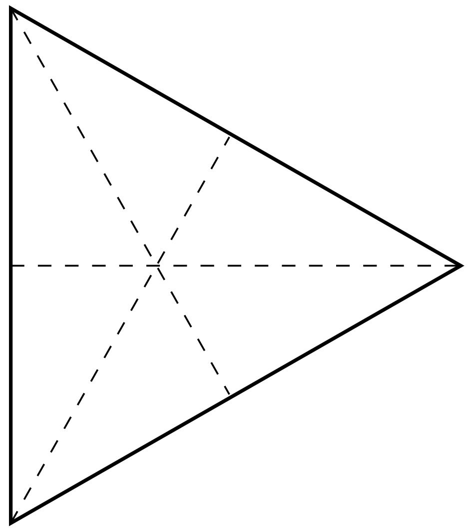
The four dotted lines XY, RS, AC and BD are known are the **lines of symmetry.** Therefore, a square has 4 lines of symmetry.

A line of symmetry divides an object or shape into similar/ identical parts, that is, one half is the mirror image of the other half.

Example 2.2.1 Find the number of lines of symmetry in the equialteral triangle below.



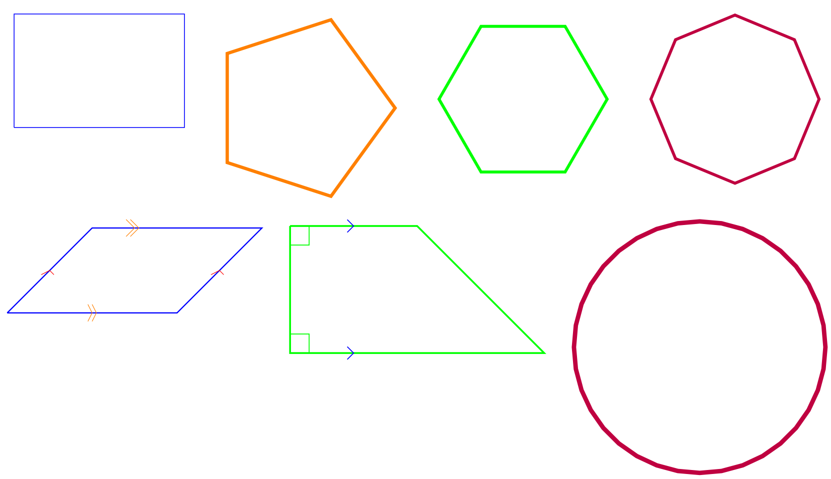
Solution. An equilateral triangle has 3 lines of symmetry each one from the vertex to the midpoint of the opposite side as shown.



□

Exercises

1. Cut out the following shapes from a paper and find the number of lines of symmetry in each.



2. Identify lines of symmetry in the alphabetical letters below

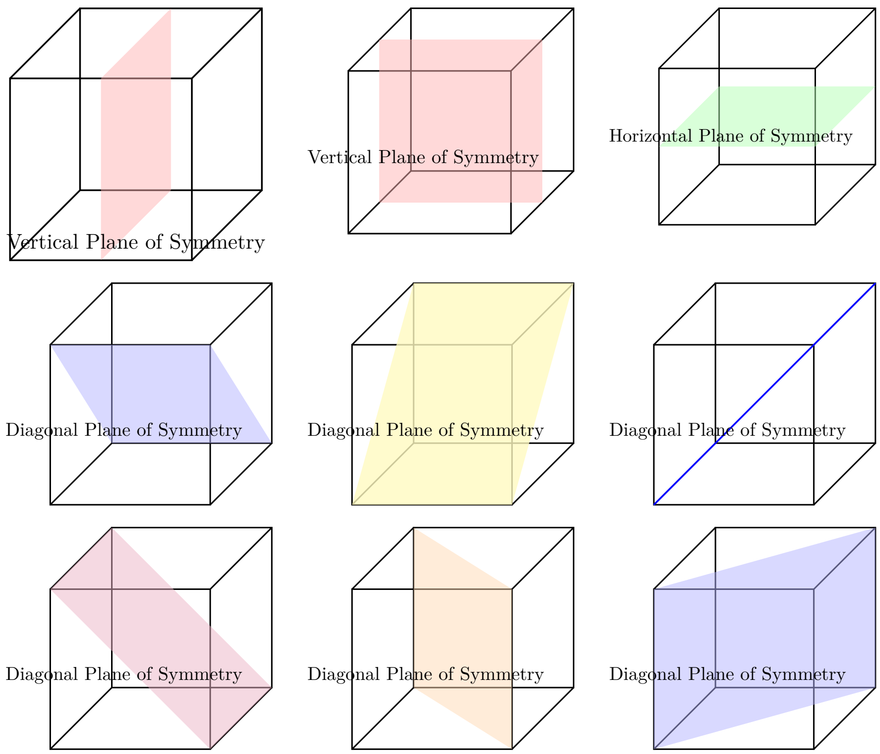
3. Identify lines of symmetry on the different objects in your classroom and at home.

Checkpoint 2.2.2 This question contains interactive elements.

2.2.2 Plane of Symmetry

Activity 2.2.2 Work in groups

Consider the activity below illustrating plane of symmetry of a cube.



There are 9 planes of symmetry in a cube as you can see in the images above.

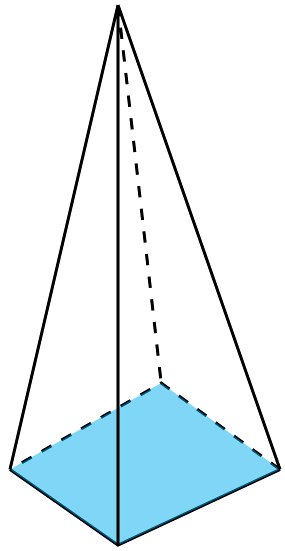
- 2 vertical planes of symmetry.
- 1 horizontal plane of symmetry.
- 6 diagonal planes of symmetry.

Key Takeaway

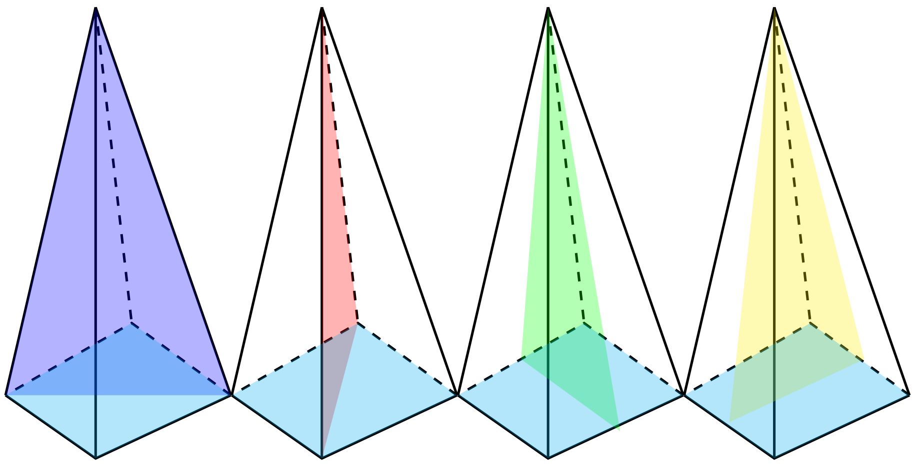
Plane of symmetry is an imaginary flat surface that divides an object into two equal halves, such that one half is the mirror image of the other half.

When you cut a banana vertically into two equal halves such that one half is the reflection of the other half, you create a plane of symmetry of the banana.

Example 2.2.3 How many planes of symmetry does a square based pyramid have?



Solution. A square based pyramid has 4 planes of symmetry as shown below.



□

Exercises

1. How many planes of symmetry are there in the following geometrical objects.

(a) Sphere.

(b) Triangular prism.

(c) Rectangular prism.

(d) Cone.

(e) Cylinder.

2. Find planes of symmetry for objects around the school.

3. Identify fruits and vegetables that have planes of symmetry.

Checkpoint 2.2.4 How many planes of symmetry are there in the following geometrical objects.

(a). Sphere.

(1) infinitely many

(2) 6

(3) 4

(4) 9

(b). Cuboid.

(1) 9

(2) 3

(3) 5

(4) 6

(c). Rectangular prism.

(1) 8

(2) 6

(3) 7

(4) 3

Answer 1. (1)
Answer 2. (2)
Answer 3. (4)

Solution. Worked solution A plane of symmetry is a plane that divides a shape into two identical halves.

A ***sphere*** has ***infinitely many planes of symmetry*** because any plane passing through its center divides it into two identical halves.

A ***cuboid*** with three different pairs of opposite rectangles has ***3 planes of symmetry***, one through the midpoints of each pair of opposite faces.

A ***rectangular prism*** with three different pairs of opposite rectangles has ***3 planes of symmetry***, one through the midpoints of each pair of opposite faces.

2.2.3 Properties of Reflection

Activity 2.2.3 Work in groups

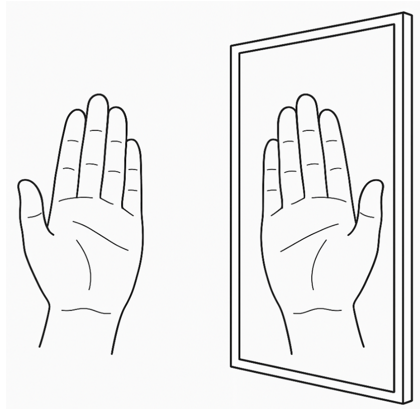
Exploring reflections with mirrors.

Needed Materials

- A small mirror.
- Marker or pencil.
- Piece of paper.

Steps.

- Hold your hand about $10\,cm$ in front of the mirror, with your palm facing the mirror. What do you notice?



- Turn your hand so that the back of your hand faces the mirror instead. Observe again. How does the reflection mimic the shape of your hand? Is there symmetry between your hand and its relection?
- Hold your hand in front of the mirror with your palm facing the mirror, then fold your fingers to make a fist. Does the reflection mimic your hand movements in the same direction?
- Slowly rotate your hand in different directions (up, down, sideways) while watching the mirror. Discuss how the mirror's reflection always mirrors the real movement.
- Move your hand closer to the mirror until it touches the mirror. What do you notice?

Key Takeaway

- The mirror does not create a random image but produces a reversed copy of the object in front of it. From your activity, you will notice that when you hold your right hand in front of the mirror, the reflection is a reverse which appers as the left hand. This phenomena is called **lateral inversion.** Lateral inversion is when a reflected object appears to be flipped along a vertical axis.
- You will also notice that the reflection mimics you hand movements when you flip, rotate it and when you fold it.
- As you move your hand closer to the mirror, you notice that the distance between reflection and the mirror reduces. When you touch the mirror, the image appears to touch your hand. Your hand and its reflection are in symmetry.

Exercises

1. Identify the parts of your body are the mirror images of one another.

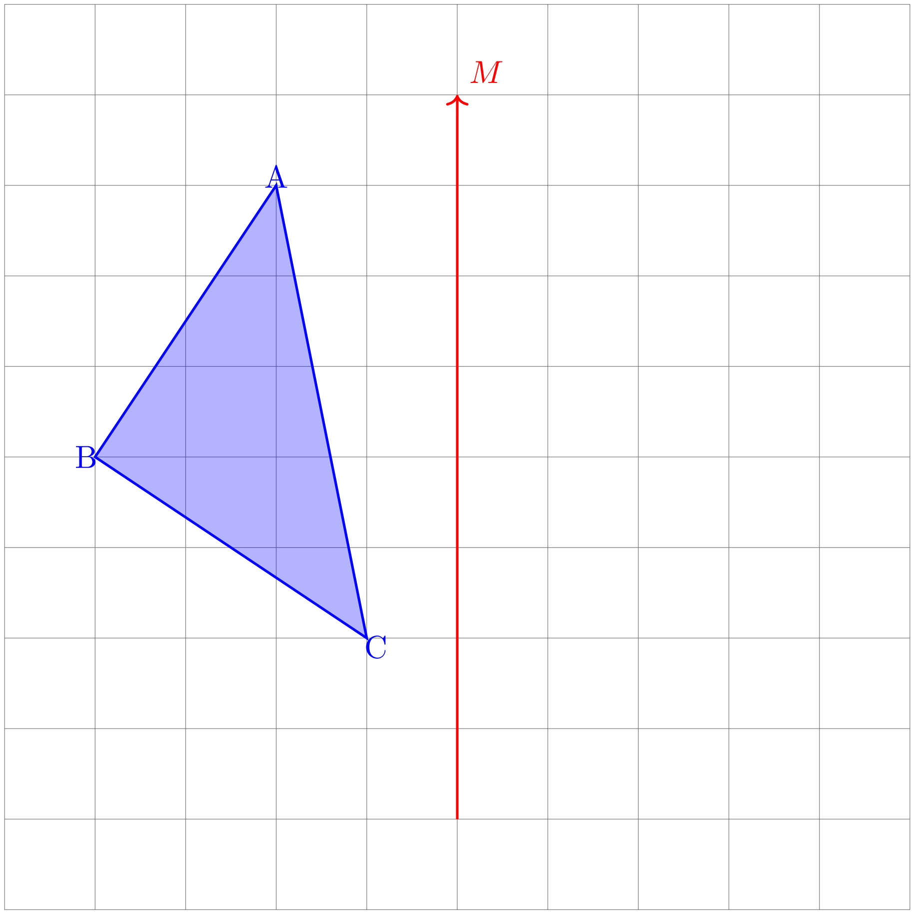
2. Draw a letter E on a paper using a marker. Show it to the mirror. Observe how the mirror flips the image.

3. What other surface reflects images other than a mirror?

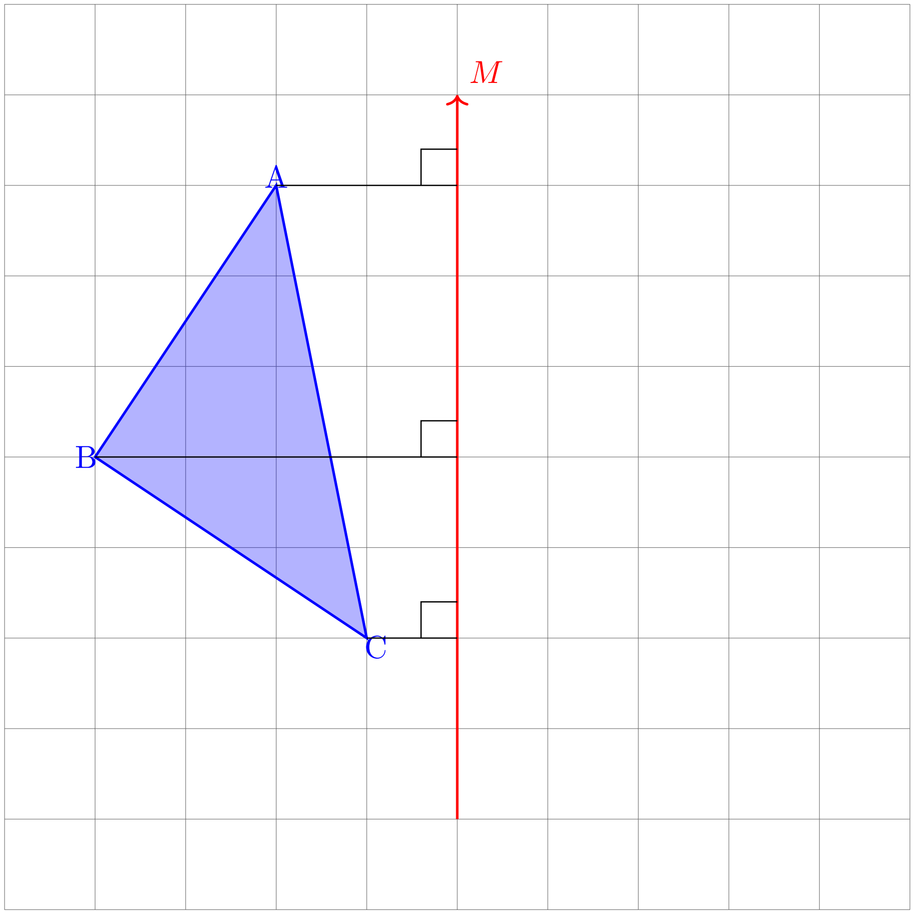
2.2.4 Reflection of Different Shapes on a Plane

Activity 2.2.4 Work in groups

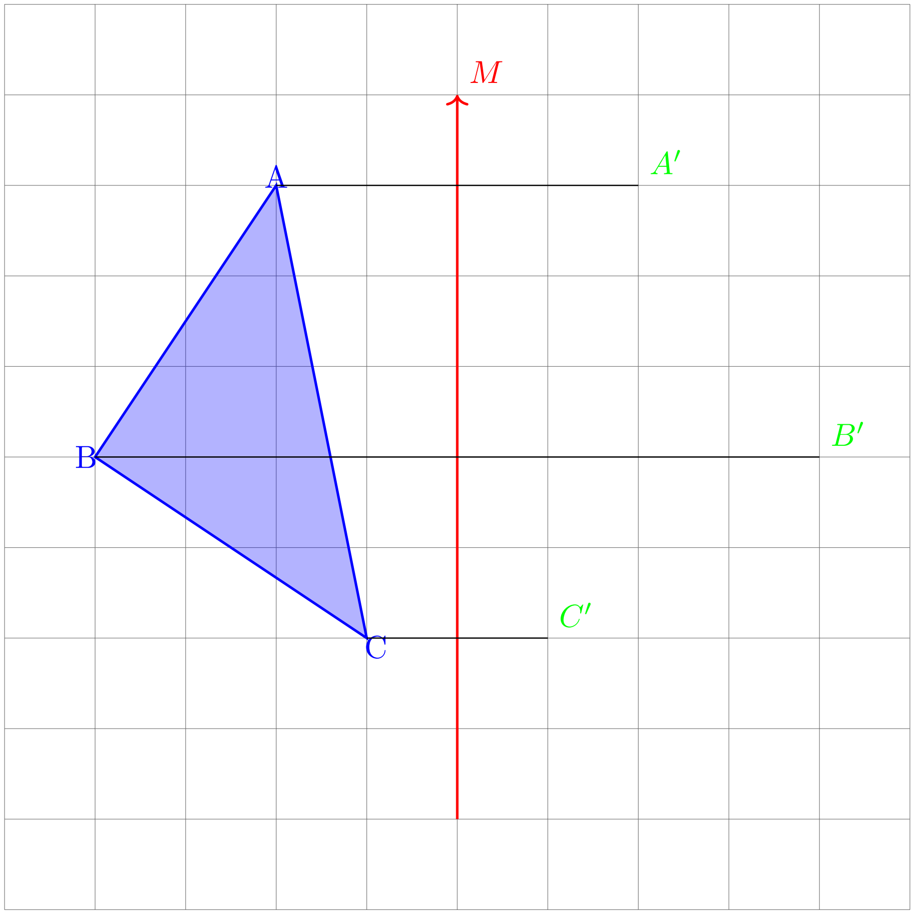
Here is a step by step approach on reflection of a triangle on a plane M.



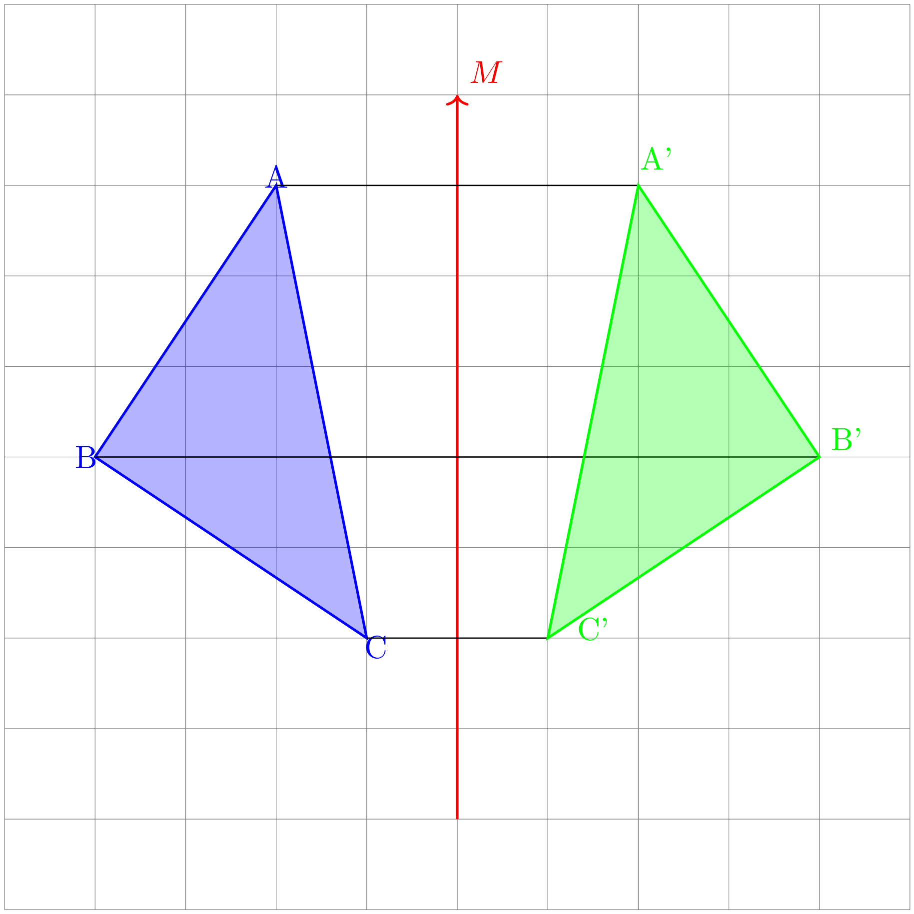
Draw a perpendicular line from vertex A to the mirror line M and measure the distance by counting the number of squares between vertex A and the mirror line M. Repeat this process for vertices B and C.



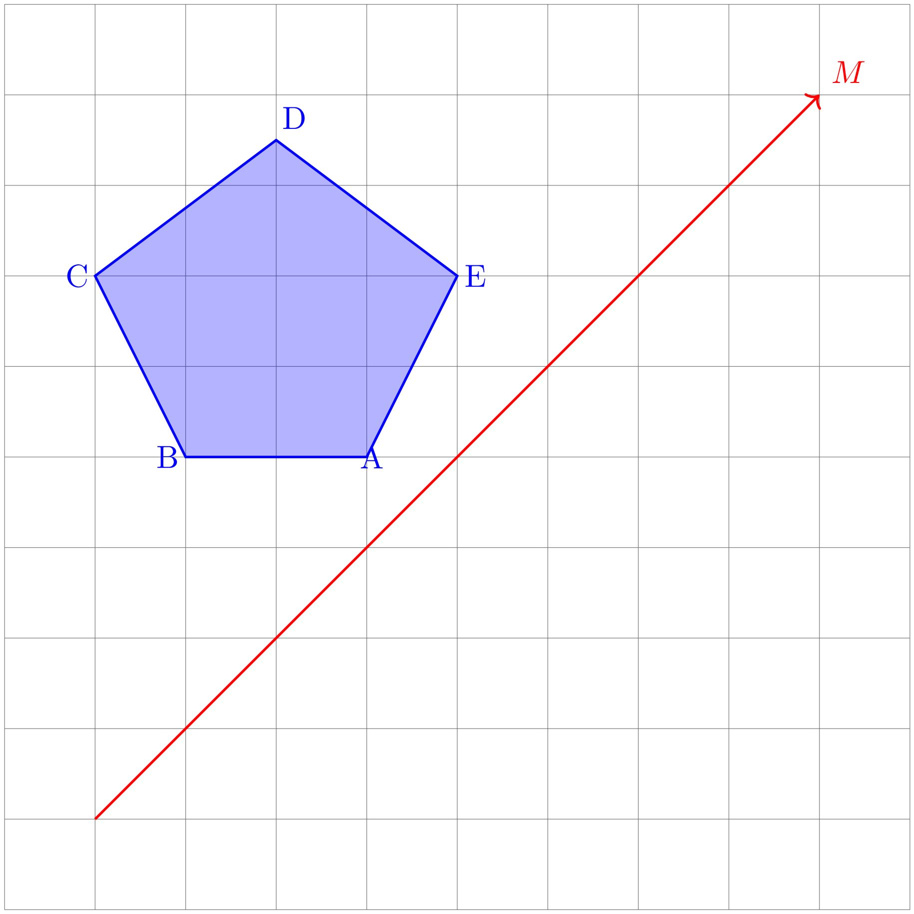
Determine the position of the reflected vertices. For vertex A the perpendicular distance between the vertex and the mirror line is 2 squares. Count 2 squares from the mirror line to the opposite side of the mirror line and mark that point as A', which is the reflected image of vertex A. Repeat the same procedure the remaining vertices B and C.



Connect the reflected vertices A', B' and C' to create the reflected image of the triangle ABC.

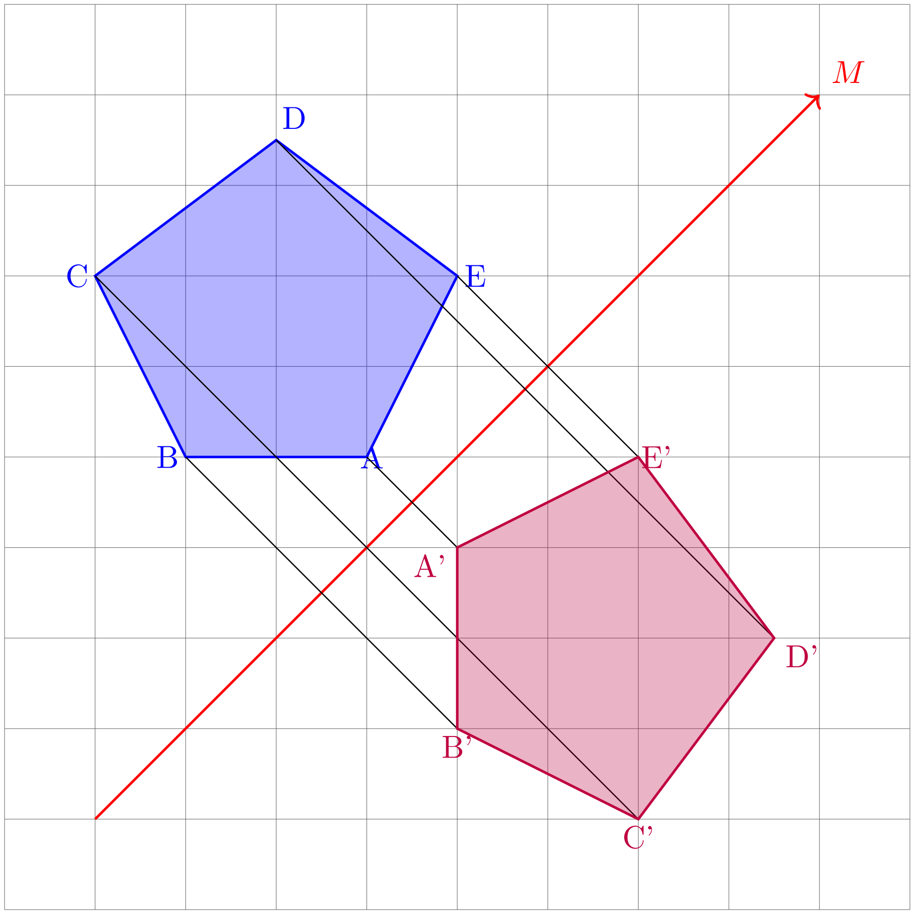


Example 2.2.5 Draw the image of the pentagon under the reflaction on the diagonal mirror line M.



Solution. To obtain the image A' of A draw a perpendicular line from A to the mirro line M, extend the line the same distance on the opposite side of the mirror line and mark the point as A'. Similarly, obtain the images B', C', D', E' the images of vertices B, C, D, E respectively.

Connect the images of the vertices to form the reflection of the pentagon.



□

Key Takeaway

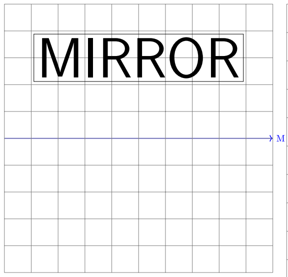
- Reflection moves the image of an object across the mirror line, that is, to the opposite side of the mirror line.
- A point on the object is the same distance as its reflection from the mirror line.
- The line connecting a point to its image is perpendicular to the mirror line. Therefore, the mirror line is the perpendicular bisector of the lines connecting the object points and the image points.

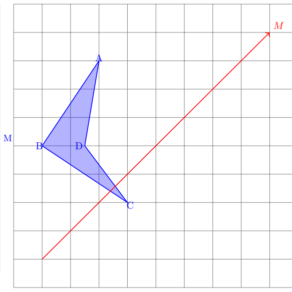
Checkpoint 2.2.6 This question contains interactive elements.

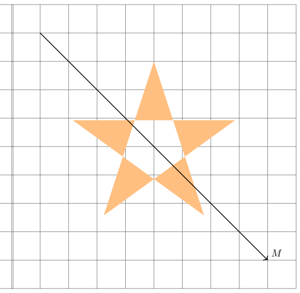
Checkpoint 2.2.7 This question contains interactive elements.

Exercises

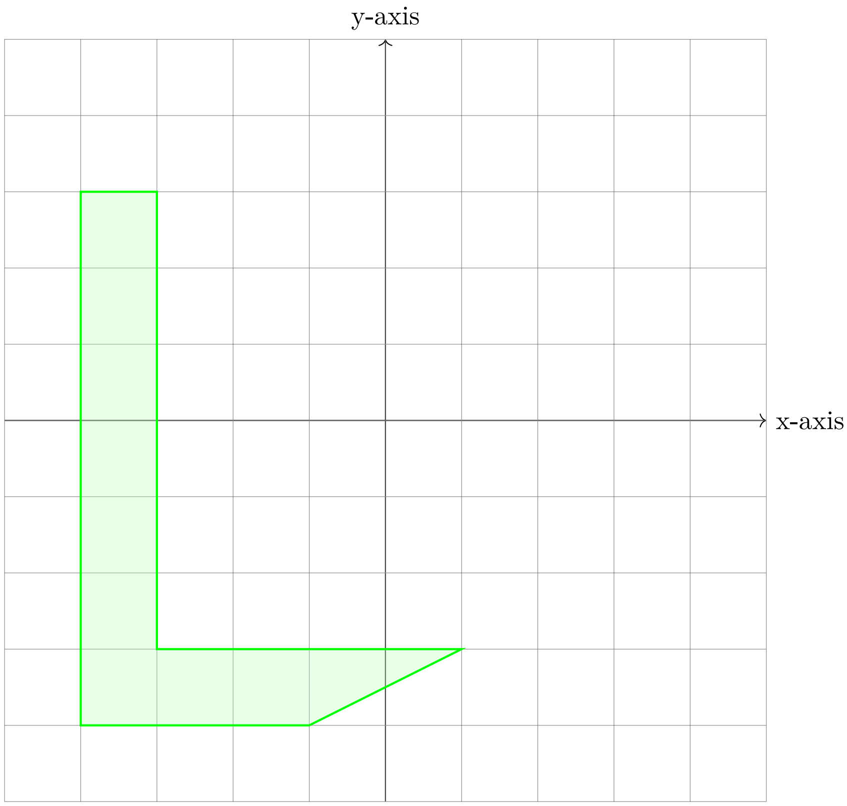
1. Copy the figures below and draw their images under the reflection on the mirror line M.







2. Reflect the object about the y-axis



3. The vertices of a polygon are given as: A(-5,5), B(-6,3), C(-5,1), D(-3,0), E(-2,2) and F(-3,4). Find the image of the polygon under the following reflection lines:

(a) $y = x$ followed by $y = 0$

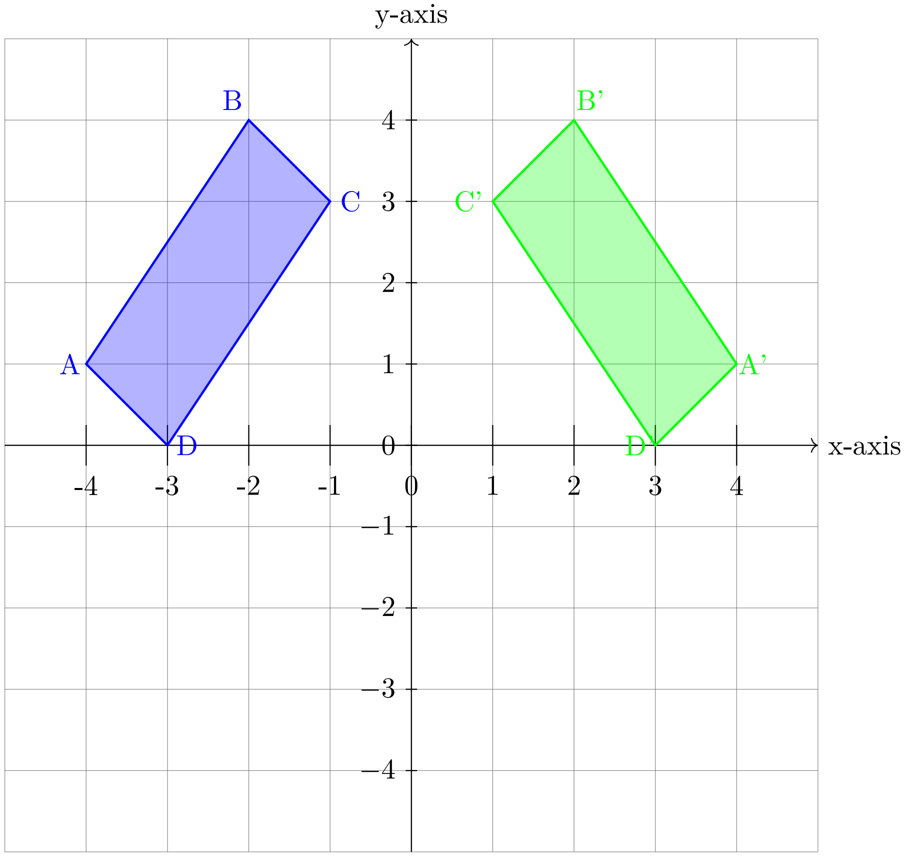
(b) $x = 0$

4. The points $A'(-4, 1)$, $B'(-2, 4)$ and $C'(-1, 3)$ are the images of points A, B and C respectively under a reflection on the line $x = -1$. Find the coordinates for points A, B and C.

2.2.5 Determining the Equation of a Mirror Line (Line of Reflection) Given an Object and its Image

Activity 2.2.5 Work in groups

Determine the line of reflection that created the reflected image below.



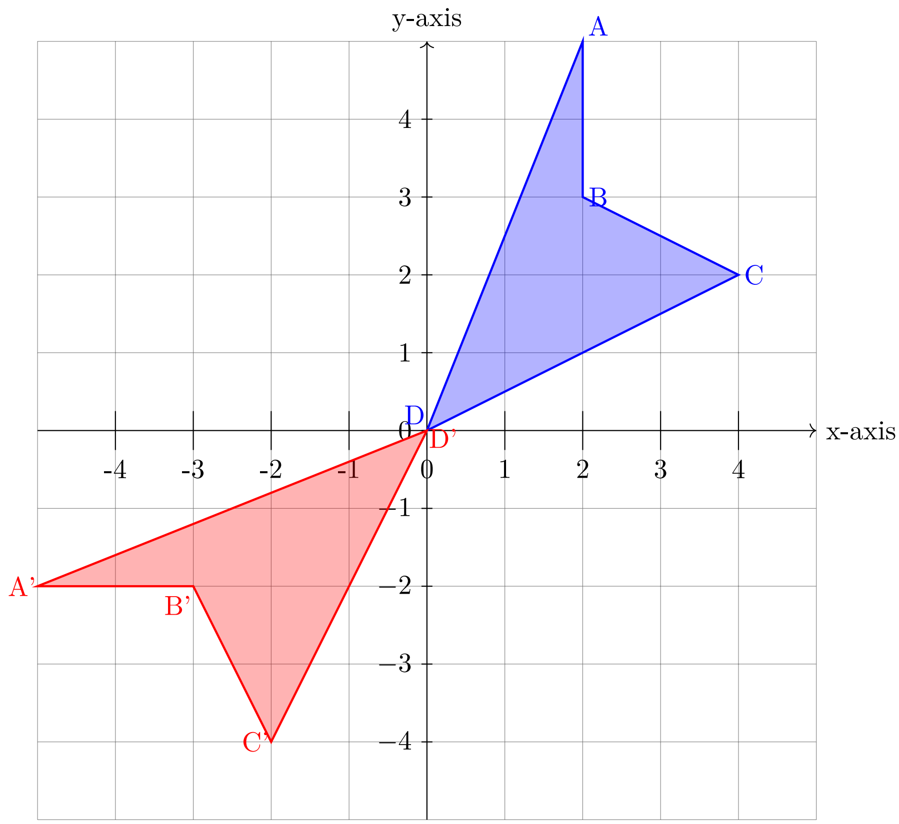
Copy the figure above on a graph paper

Fold your graph paper such that the points of the objects match with their respective images. Where does the fold line appear?

Key Takeaway

- You notice that the fold line appears exactly on the y-axis. Therefore, the line of reflection is the y-axis.
- A line of reflection can be defined with there equation. From the activity, the equation of the line of reflection is $x = 0$.

Example 2.2.8 Determine the equation of the line of reflection.



Solution. The coordinates of D and D' are at $(0, 0)$, tells you that the line of reflection passes through $(0, 0)$.

Connect point C to C' with a line. The line of reflection is the perpendicular bisector of C and C'.

From the properties of reflection, the distance from the object to the mirror line is the same as that of mirror line to the image. Therefore, the line of reflection passes through the midpoint of the line connecting C to C'.

Coordinates for C is $(4, 2)$ and that of C' is $(-2, -4)$. The mid point of line CC' is:

$$\left(\frac{4 + -2}{2}, \frac{2 + -4}{2}\right) = (1, -1)$$

Since you know that the line of reflection passes through $(0, 0)$ and $(1, -1)$, the gradient of the reflection line is:

$$m = \frac{-1 - 0}{1 - 0} = -1$$

Therefore taking points (x, y) and $(1, -1)$, the equation of the line of reflection is :

$$y - y_1 = m(x - x_1)$$

$$y - -1 = -1(x - 1)$$

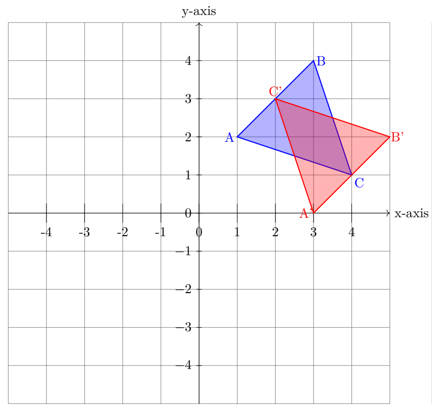
$$y + 1 = -x + 1$$

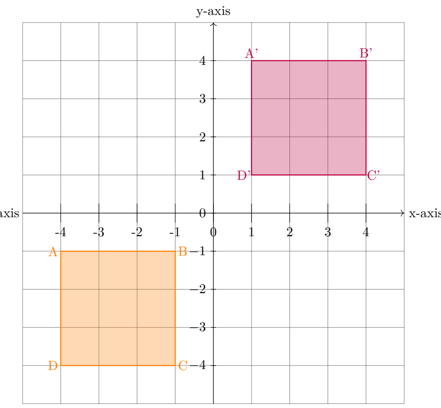
$$y = -x$$

□

Exercises

1. Determine if the transformation is a reflection. If it is a reflection, what is the equation of the mirror line?





2. The vertices of a triangle are $A(1, 2)$, $B(3, 4)$ and $C(5, 4)$. The vertices of the image are $A'(1, -2)$, $B'(3, -4)$ and $C'(5, -4)$. Find the equation of the line of reflection.

3. The vertices of a a letter V are $P(-3, 4)$, $Q(-3, 2)$ and $R(-1, 2)$. The vertices of the image are $P'(-1, 2)$, $Q'(-1, 0)$ and $R'(1, 0)$. Find the equation of the line of reflection.

4. $O(0, 0)$ is the centre of a circle of radius $2\,cm$. If $O'(2, 0)$ is the reflection of the centre of the circle, find the equation of the line of reflection.

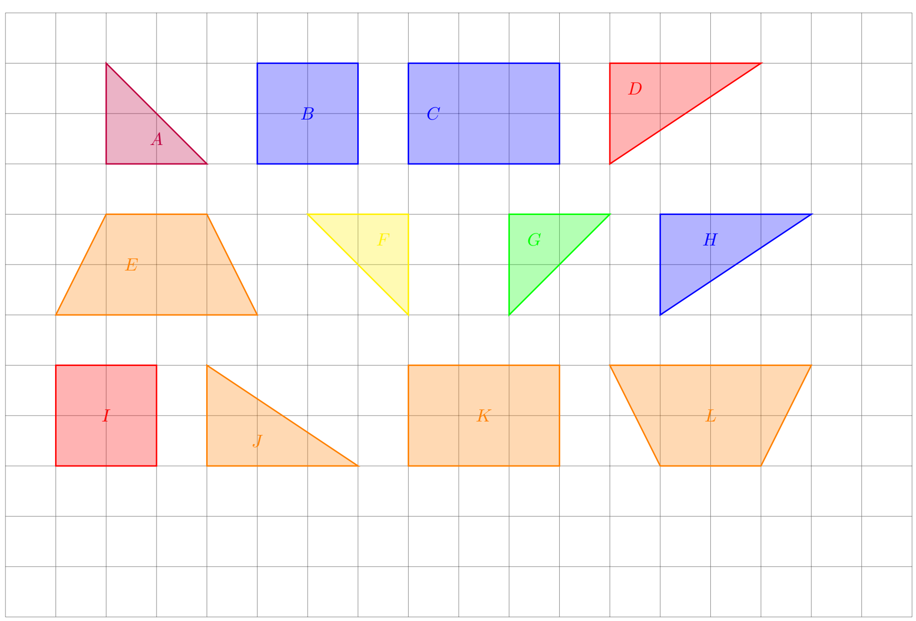
Checkpoint 2.2.9 This question contains interactive elements.

Checkpoint 2.2.10 This question contains interactive elements.

2.2.6 Congruence

Activity 2.2.6 Work in groups

Trace and cut out the following shapes on a piece of paper.



Match two shapes by aligning them directly on top of each other. Discuss how the sides and angles perfectly overlap.

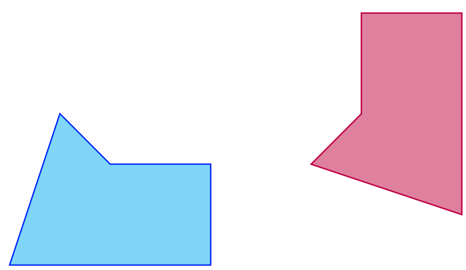
Flip one shape (like turning it over) and try to align it with another shape. Discuss how the shapes still match in size and angles, even though one is flipped.

- Identify the figures with the same shape.
- Identify the figures with the same size.
- Identify figure with same size and shape.

Key Takeaway

- Figure with the same size and shape are said to be **congruent.**
- The symbol for congruence is $\cong$. For example, in your activity, $B \cong I$.
- Also, B fits directly on I without flipping, thus B and I are said to be **directly congruent.** Identify other figures that are directly congruent.
- E and L do not fit directly but when you flip **(lateral inversion)** figure L and fit it on E, they align. This is called **opposite congruence.** Identify other figure that have opposite congruence from your activity.
- Figures with the same shape but different size are said to be **similar.**
- Congruence can also be applied to line segments and angles.

Example 2.2.11 Here is an example of congruent polygons.



□

Checkpoint 2.2.12 What type of triangle(s) would be both directly and oppositely congruent to its image after a reflection ?

(1) isosceles triangle

(2) right angle triangle

(3) scalene triangle

(4) equilateral triangle

Answer. $(1), (4)$

Solution. Explanation A reflection reverses orientation. Most triangles are therefore only *oppositely congruent* to their mirror images.

An ***isosceles triangle*** has a line of symmetry, so under reflection it can coincide with its image.

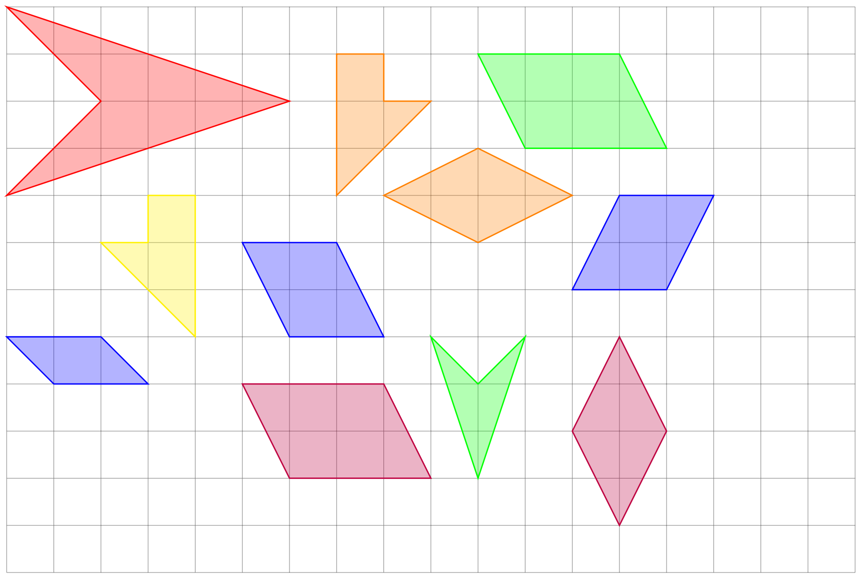
An ***equilateral triangle*** is even more symmetric: all sides and angles are equal, so it is completely indistinguishable from its reflection.

For this reason, both ***isosceles*** and ***equilateral*** triangles are accepted here.

Exercises

1. Identify figures that are:

- Directly congruent.
- Oppositely congruent.
- Similar.



2. Identify object around your school that have axis of symmetry and state the order of rotational symmetry.

2.2.7 Congruence Tests for Triangles

Activity 2.2.7 Work in groups

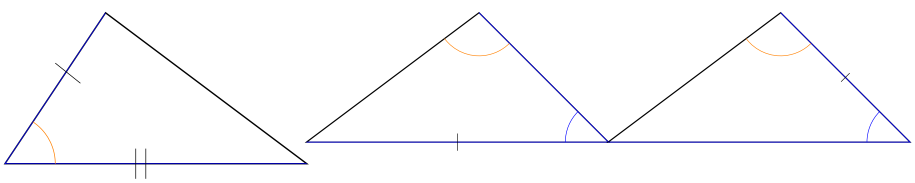
Conditions for Congruence in Triangles

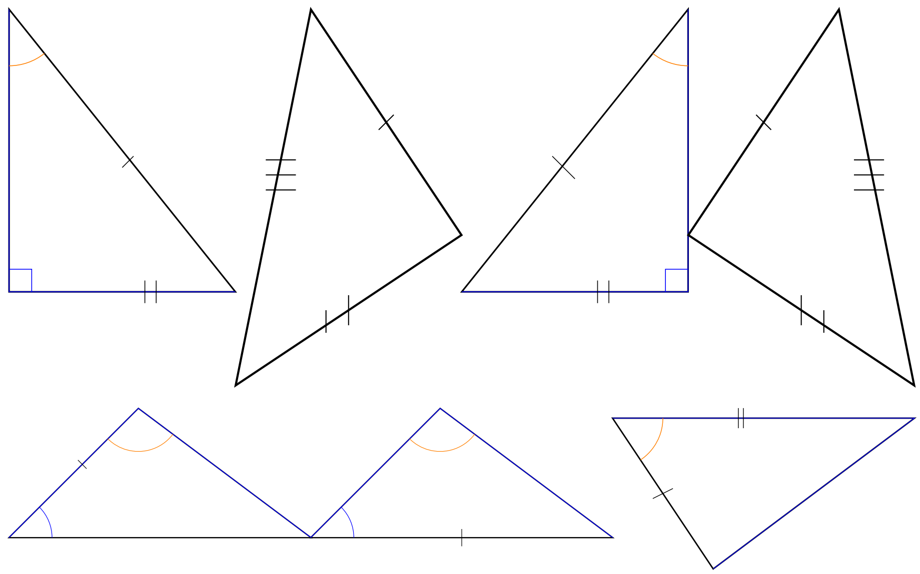
Materials

- Construction paper
- Pencil, ruler, protractor

Instructions

Trace the following triangles on a construction paper.





Identify pairs of congruent triangles.

From the pairs of congruent triangles you have identified, which pairs fit the following criteria:

1. The three sides of one triangle is equal to the three sides of the corresponding triangle.
2. Two sides and an included angle of one triangle is equal to the two corresponding sides and the included angle of the other triangle.
3. One side and two included angles of one triangle is equal to the corresponding side and the two included angles of the other triangle.
4. One side and the hypotenuse of a right-angled triangle are equal to the hypotenuse and the corresponding side of another right-angled triangle.

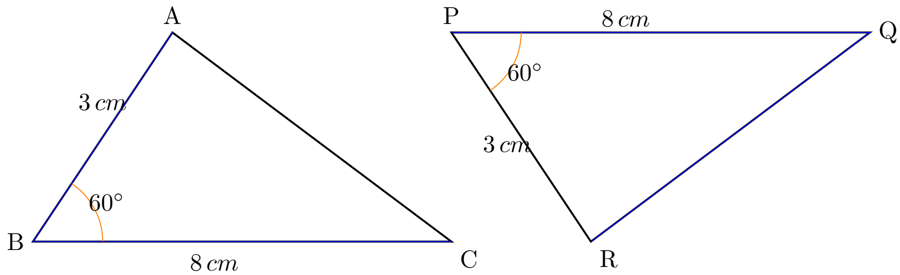
Key Takeaway

Congruence in triangles dependS on the measure of the sides and angles. Two triangles are said to be congruent if a pair of the corresponding sides and corresponding angles are equal.

Criteria for congruence tests in triangles include:

- **Side-side-side (SSS):** the three sides of one triangle is equal to the three sides of the corresponding triangle.
- **Side-angle-side (SAS):** two sides and an included angle of one triangle is equal to the two corresponding sides and the included angle of the other triangle.
- **Angle-side-angle (ASA):** one side and two included angles of one triangle is equal to the corresponding side and the two included angles of the other triangle.
- **Right angle-hypotenuse-side (RHS):** one side and the hypotenuse of a right-angled triangle are equal to the hypotenuse and the corresponding side of another right-angled triangle.
- **Angle-angle-side (AAS):** one side and two included angles of one triangle is equal to the corresponding side and the two included angles of the other triangle.

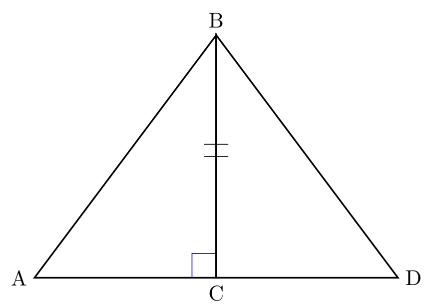
Example 2.2.13 Check if the triangles below are congruent and state the test of congruence criterion.



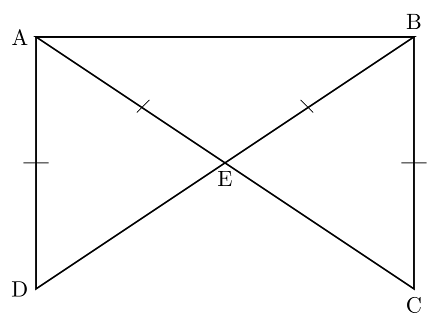
Solution. From the figure, identify corresponding sides and angles.
Sides $AB = PR = 3\,cm$ and $BC = PQ = 8\,cm$
$\angle B = \angle P = 60°$.
Therefore, $\Delta ABC \cong \Delta PQR$ by **SAS** criterion ☐

Exercises

1. Check if the triangles below are congruent and state test of congruence criterion.



2. Show that $\Delta ABC \cong \Delta ADB$ if $AD = AE = BE = BC$.



3. $A(0,4)\,B(-3,0)$ and $C(0,2)$ are the coordinates of ΔABC. Reflect the triangle over mirror line $x = 0$. Prove that the triangle and its image are congruent and state the test of congruence criterion.

4. Construct an equilateral triangle UVW with sides $6\,cm$. X is the midpoint of UW and VX is perpendicular to UW. Show that $\Delta UVX \cong \Delta VWX$. State the test of congruence criterion.

Checkpoint 2.2.14 An error occurred while processing this question.

Checkpoint 2.2.15 This question contains interactive elements.

2.3 Rotation

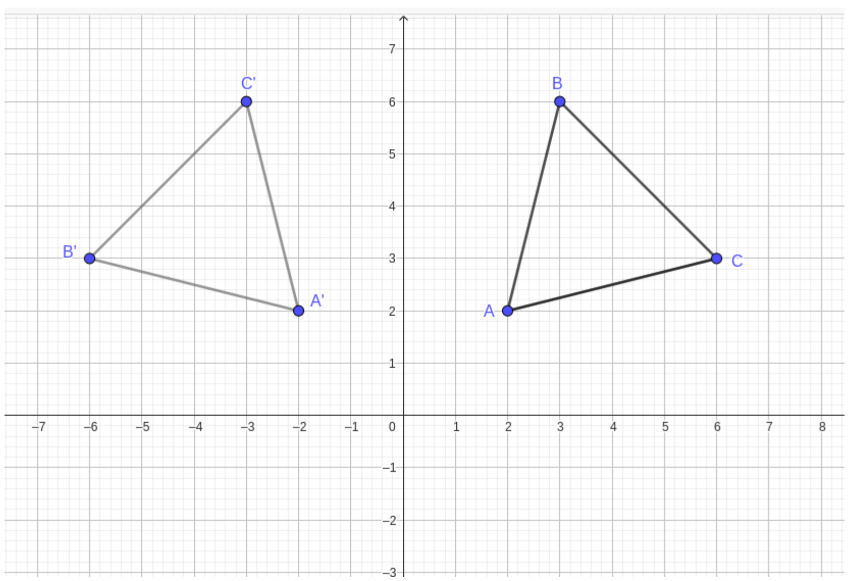
- Have you ever watched the hands of a clock move, seen a merry-go-round spin, or noticed the Earth turning? These all involve rotation—the spinning of an object around a fixed point or axis. Rotation is everywhere!
- When you ride a merry-go-round or ride a bicycle, the wheels rotate, making everything move smoothly. The Earth itself rotates, which gives us day and night. Even when you stir your drink, you're creating a small rotation!
- Rotation is a transformation that moves an object around a fixed point but its size and shape are not changed. Rotations are sometimes called turns. The point around which a rotation occurs is called the centre of rotation, and the distance a shape turns is called the angle of rotation
- In this lesson, we will explore the beauty of rotation, understand its principles, and see how it plays a role in everything.

2.3.1 Properties of Rotation

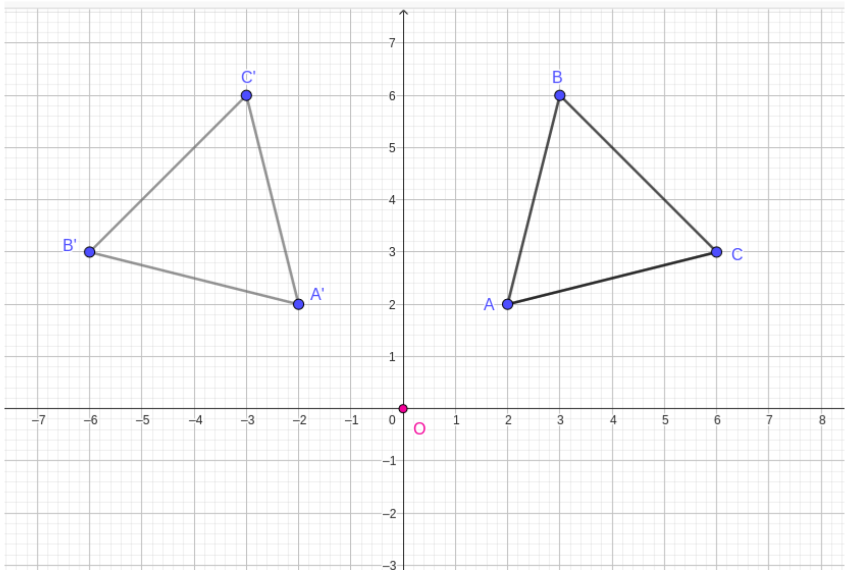
Activity 2.3.1 ***Work in pairs***

What you need: Graph paper, a ruler, a protractor and a pencil.

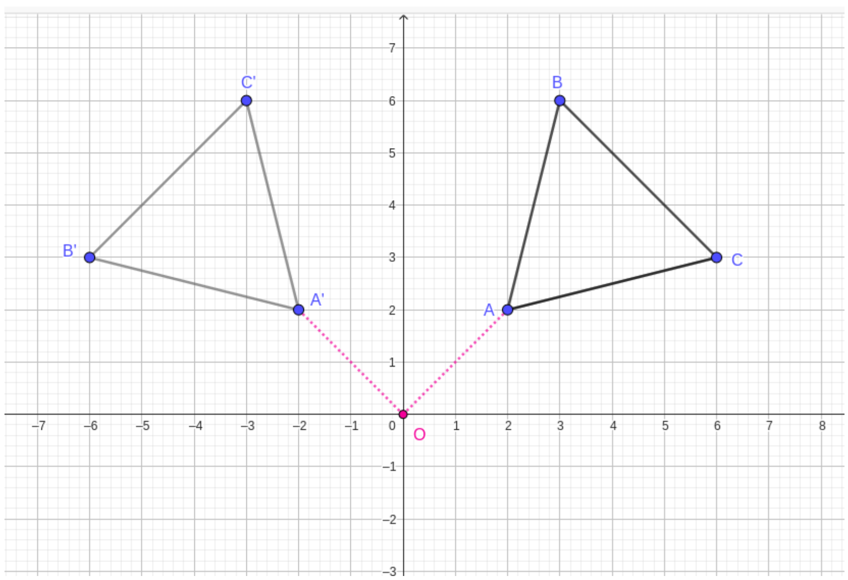
(a) On a graph paper, draw triangle ABC and its image, triangle $A'B'C'$ as shown in the figure below.



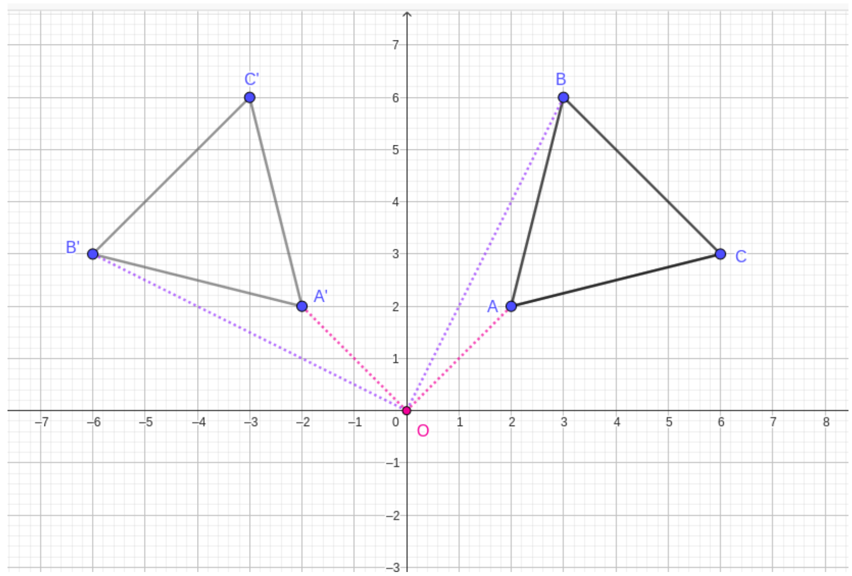
(b) Pick a point on the graph to act as the centre of rotation. Mark this point O



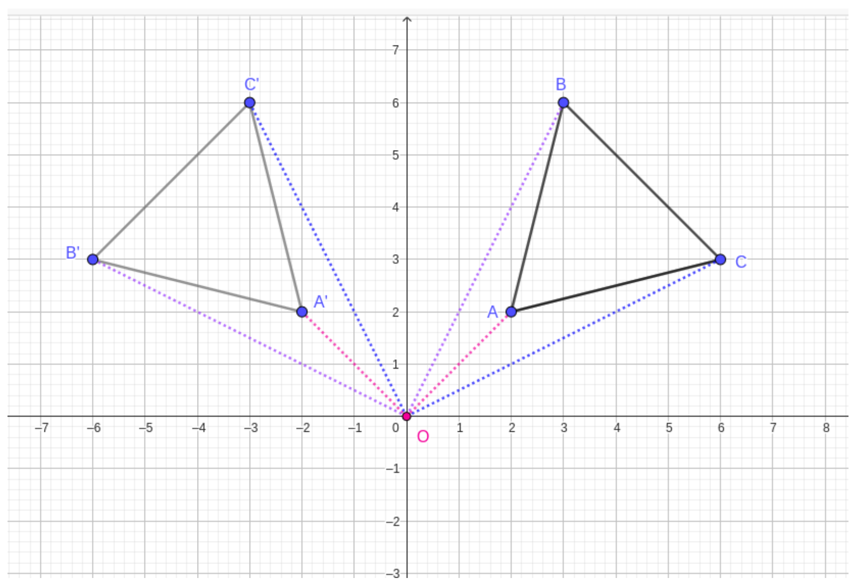
(c) Using a ruler, draw a straight line from point A to O and also from point A' to O as shown below. Measure the distance OA and OA' and record your results. What do you notice?



(d) Similarly, draw a straight line from point B to O and also from point B' to O. Measure the distance OB and OB' and record your results. What do you notice?



(e) Finally, draw a straight line from point C to O and also from point C' to O. Measure the distance OC and OC' and record your results. What do you notice?



(f) Now using a protractor, measure $\angle AOA'$, $\angle BOB'$ and $\angle COC'$ and record your results. What do you notice?

Key Takeaway

The distance $AO = A'O$, $BO = B'O$ and $CO = C'O$

The distance from a point to the centre of rotation is the same as the distance from the image of that point to the centre of rotation.

$\angle AOA' = \angle BOB' = \angle COC' = 90°$

The angle of rotation is the same for all points in the shape.

In this case, Point O is the centre of rotation and angle $90°$ is the angle of rotation.

Note:

- A rotation in the anticlockwise direction is taken to be positive i.e a rotation of $45°$ anticlockwise is $+45°$

- A rotation in the clockwise direction is taken to be negative i.e a rotation of 45° clockwise is −45°
- In general, for a rotation to be completely defined, the centre and angle of rotation must be stated.

Example 2.3.1

The coordinates of the vertices for triangle PQR that can be graphed in the coordinate plane are $(-8,-6)$, $(-2,-6)$ and $(-5,-3)$ as shown below. The triangle is rotated through 90° in a clockwise direction about the origin to produce triangle $P'Q'R'$.Copy the figure and draw triangle $P'Q'R'$

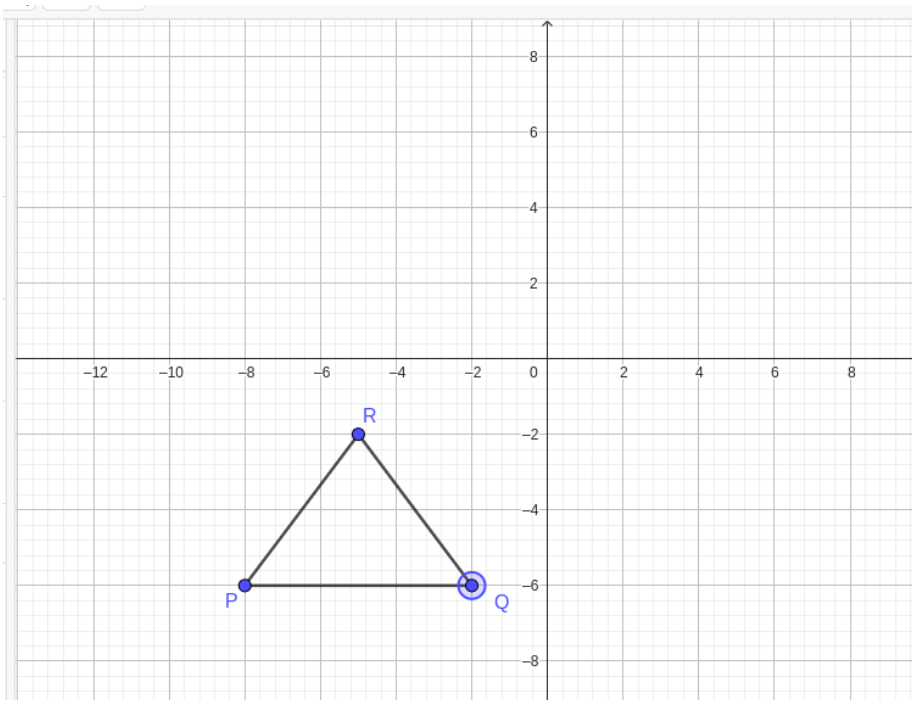


Figure 2.3.2

Solution.

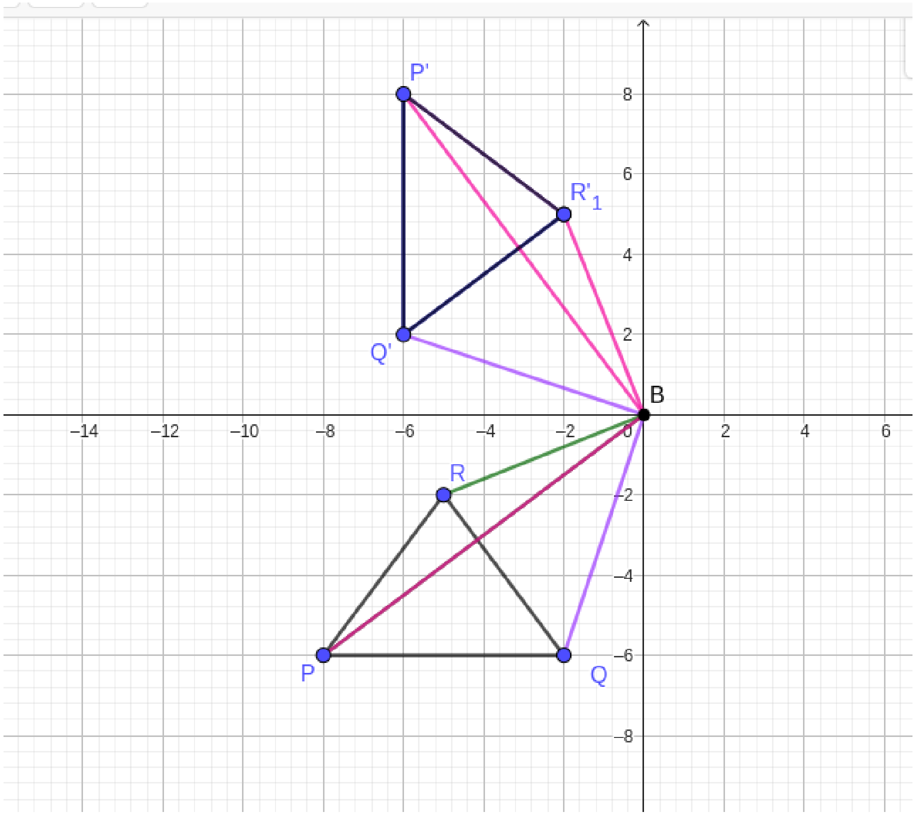


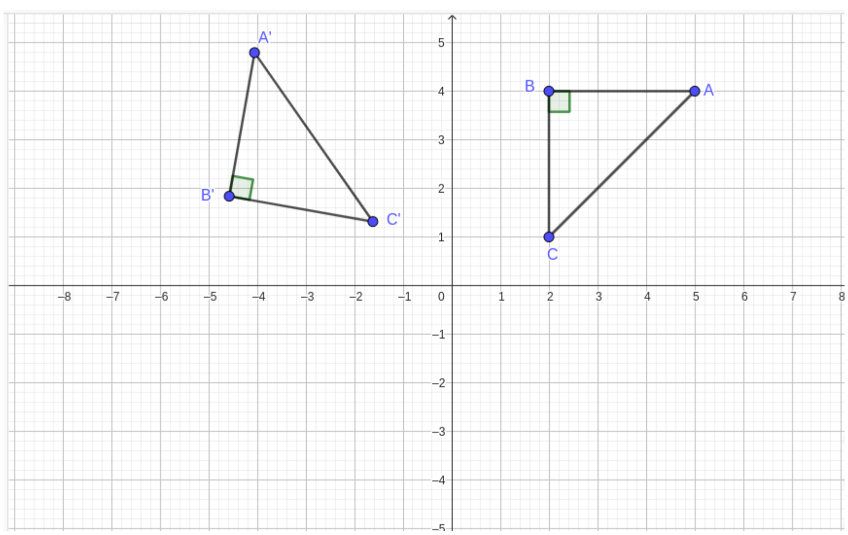
Figure 2.3.3

□

2.3.1.1 Centre and Angle of Rotation

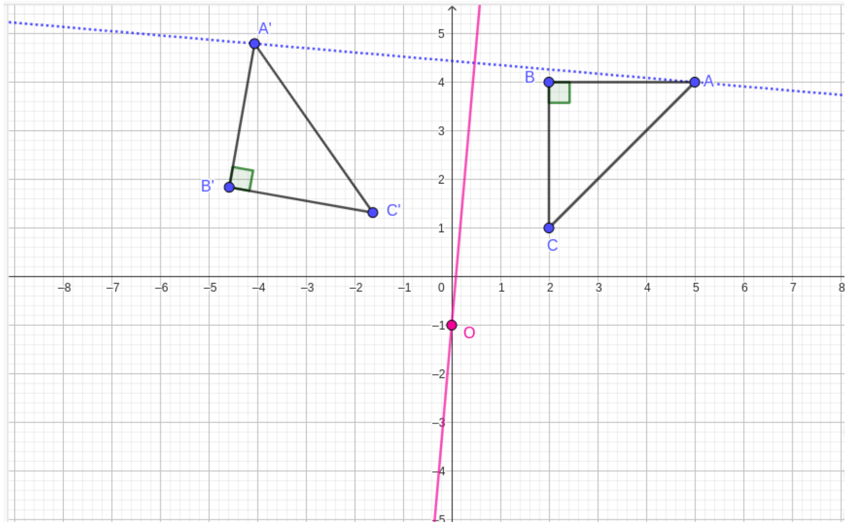
Activity 2.3.2 Work in pairs.
What you need: Graph paper, a ruler, a protractor and a pencil

1. On a piece of graph paper draw triangle ABC and its image $A'B'C'$ as shown in the figure below.

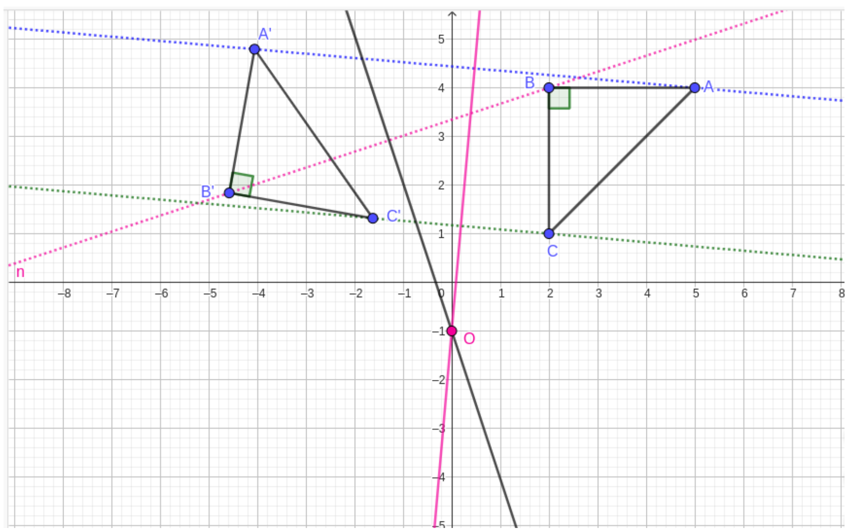


2. Join point A to A' and construct a perpendicular bisector to AA' as

shown below;

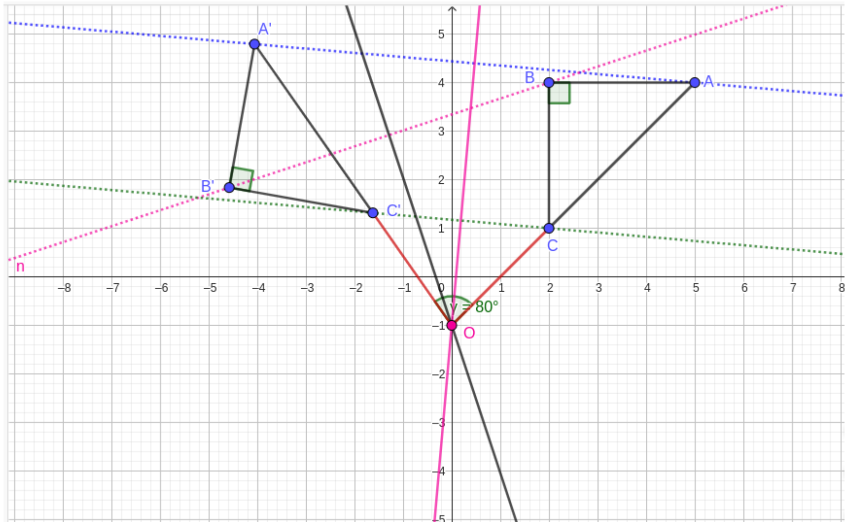


3. Similary Join point B to B' and C to C' and construct a perpendicular bisector to BB' and CC' as shown below;



4. Now join point O to C' and C and measusure $\angle COC'$

5. Similary join point O to B' and B and O to A' and $A$and measusure $\angle BOB'$ and $\angle AOA'$. What do you notice?



6. Share your work with other leaners in class

Key Takeaway

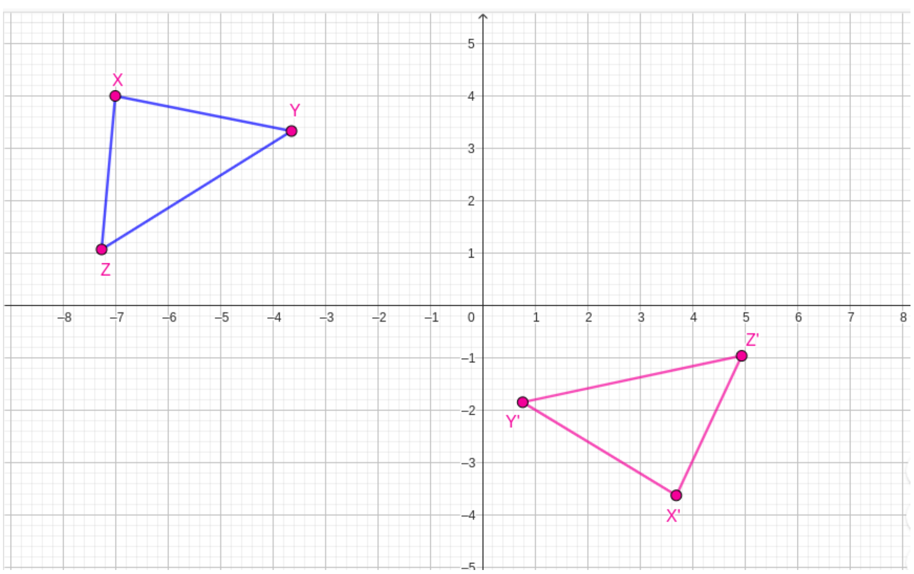
Triangle $A'B'C'$ is the image of triangle ABC after a rotation.The centre and angle of rotation can be found by drawing the perpendicular bisectors of the lines between two sets of points, C and C' and B and B' or A and A'

The point where two perpendicular bisectors intersect is called **the centre of rotation.** To find **the angle of rotation**, join C' and C to the centre of rotation and measure the angle between these lines.

$\angle COC' = \angle BOB' = \angle AOA'$

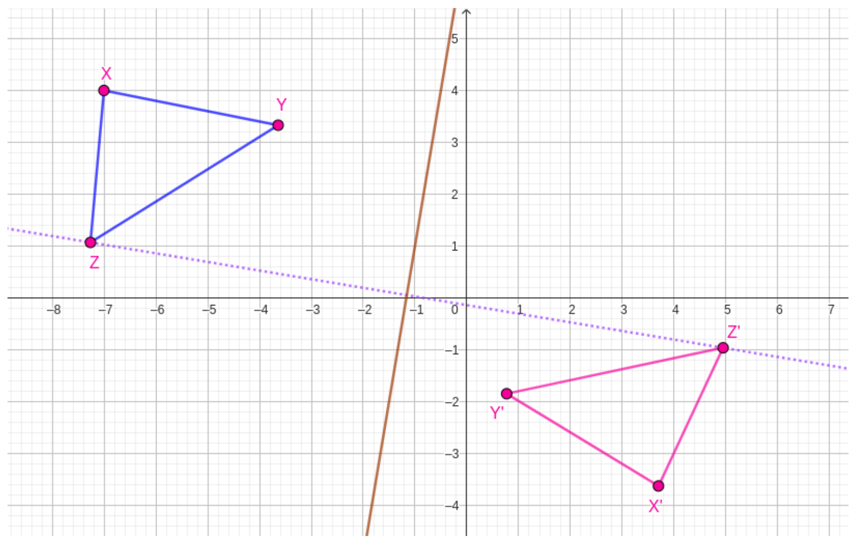
Example 2.3.4

In the figures below, the triangle $X'Y'Z'$ is the image of triangle XYZ after rotation. Find the centre and angle of rotation

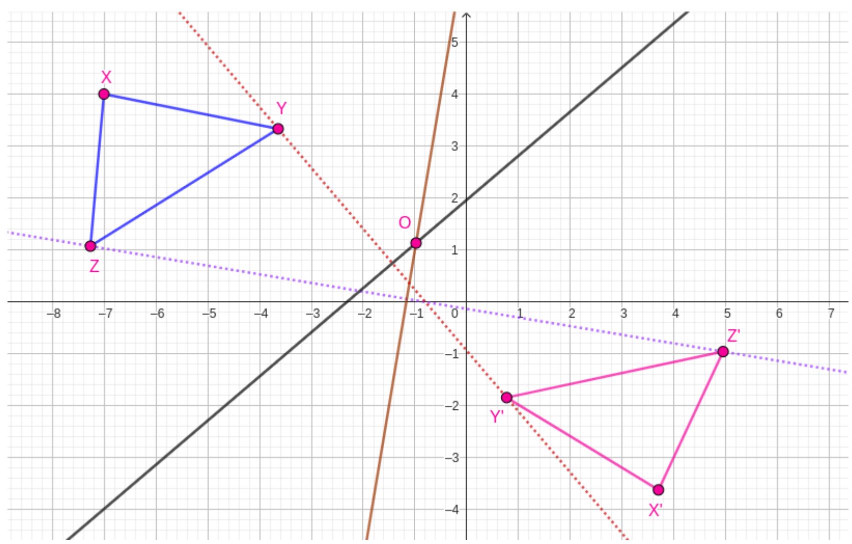


Solution. In order to determine the centre and angle of rotation we have to follow the following steps:

1. Join point Z to Z' and construct a perpendicular bisector to ZZ'' as shown below



2. Also join point Y to Y' and construct a perpendicular bisector to YY' as shown below. Mark the point of intersection of perpendicular bisectors O

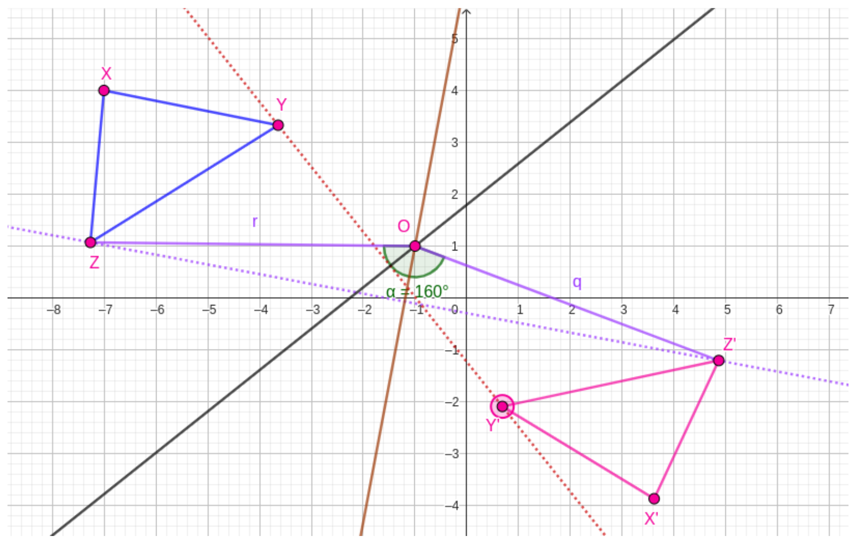


3. Similary you can join X to X' and construct a perpendicular bisector to XX'

   Note You can use only two points.

   The point where perpendicular bisectors intersect is the centre of rotation.

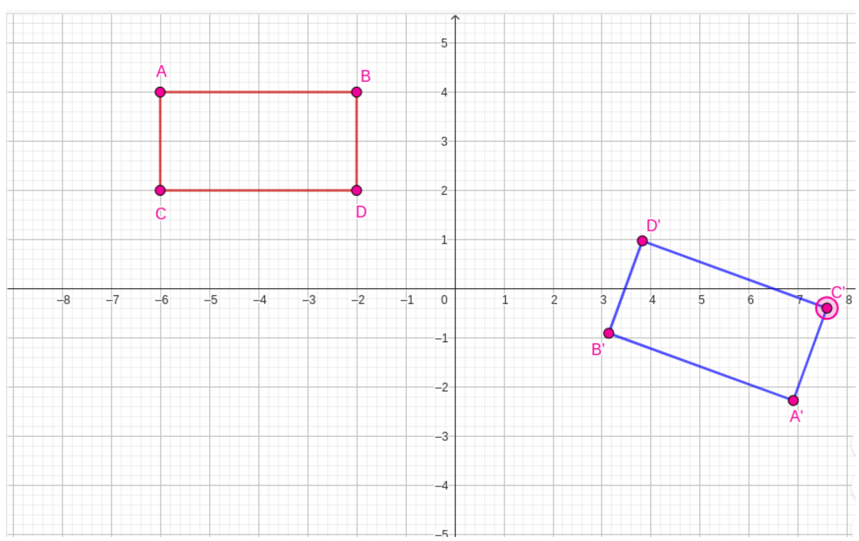
4. Now join Z and Z' to the centre of rotation O. Measure $\angle, ZOZ'$ using a protractor.



Centre of rotation $= (-1, 1)$
Angle of rotation $= -160°$ since rotation is done in a clockwise direction
□

Exercises

1. In the figure below, rectangle $A'B'C'D'$ is the image of rectangle $ABCD$ under a rotation, centre O



(a). By construction,find and label the centre O of roration.
(b) Determine the angle of rotation.

Checkpoint 2.3.5 This question contains interactive elements.

2.3.2 Rotation on Different Planes

Rotation on different planes" refers to the concept of rotating an object or point around various axes within different planes in a three-dimensional space.

2.3.2.1 Rotation in the Cartesian Plane

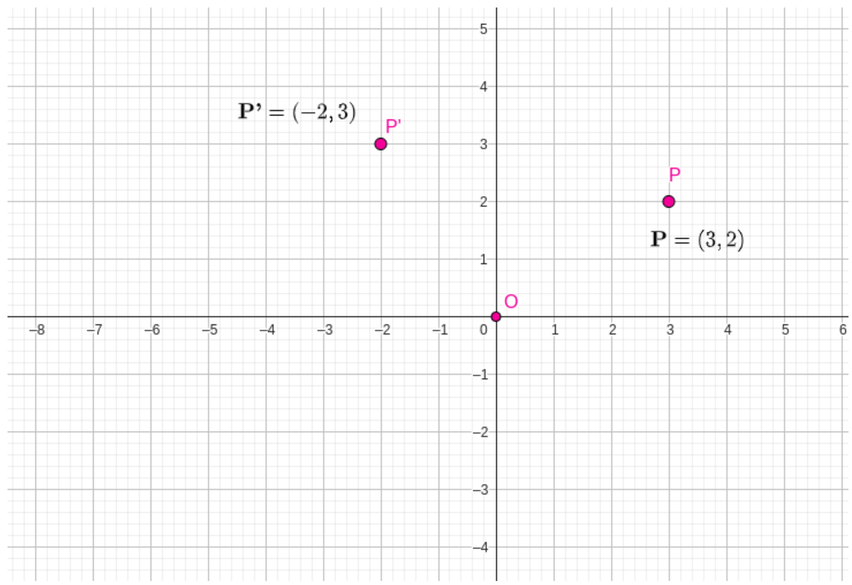
Activity 2.3.3 ***Work in pairs***

(a) Draw a large $X-$ axis (horizontal) and $Y-$ axis (Vertical) on a graph of paper. Mark the origin $(0,0)$ where the two axis meet.

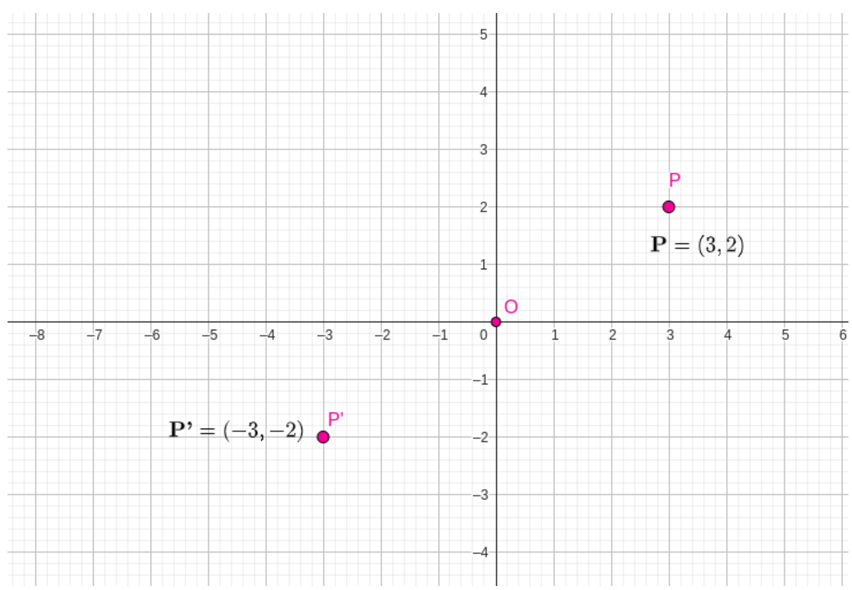
(b) Pick any point $P\,(x,y)$ and plot this point on the plane and label it for example let;s use $P\,(3,2)$.

(c) Rotate the point in a counterclockwise direction around the origin with different angles as shown below:

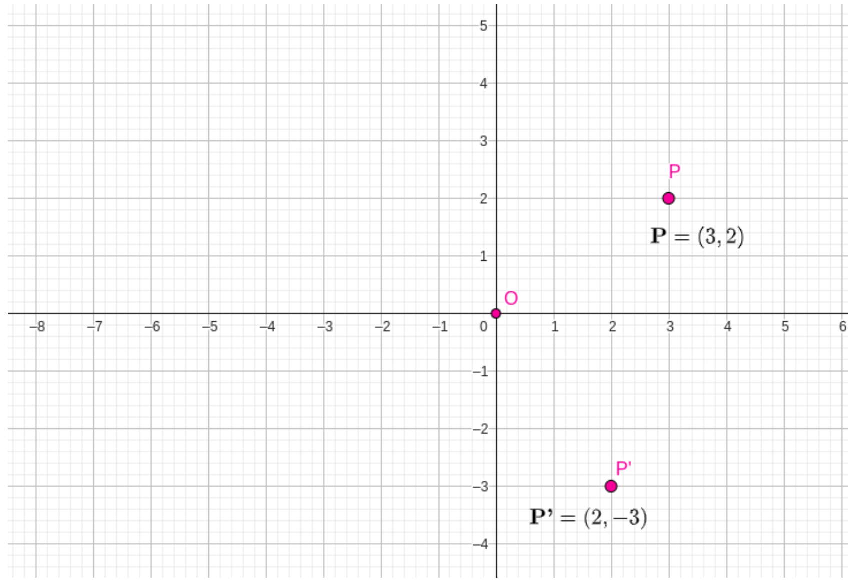
(i)**+90°**



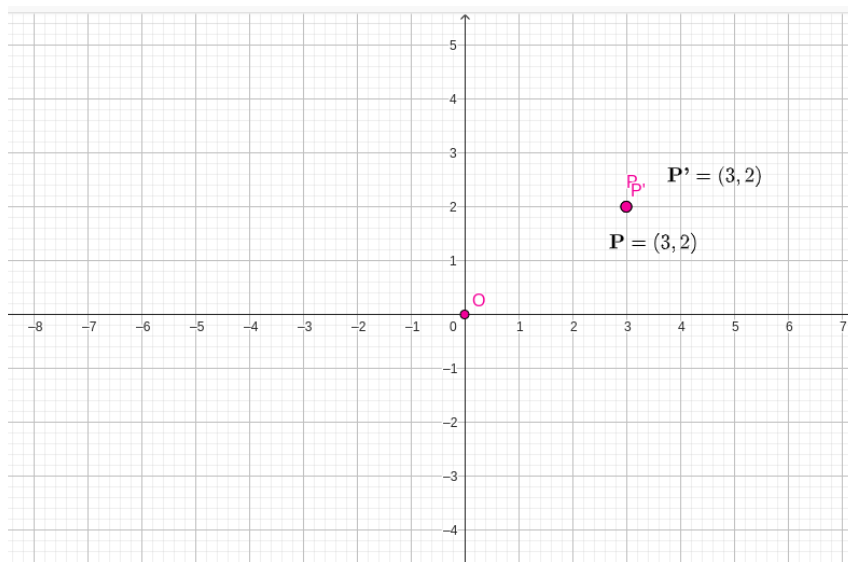
(ii)**+180°**



(iii)**+270°**

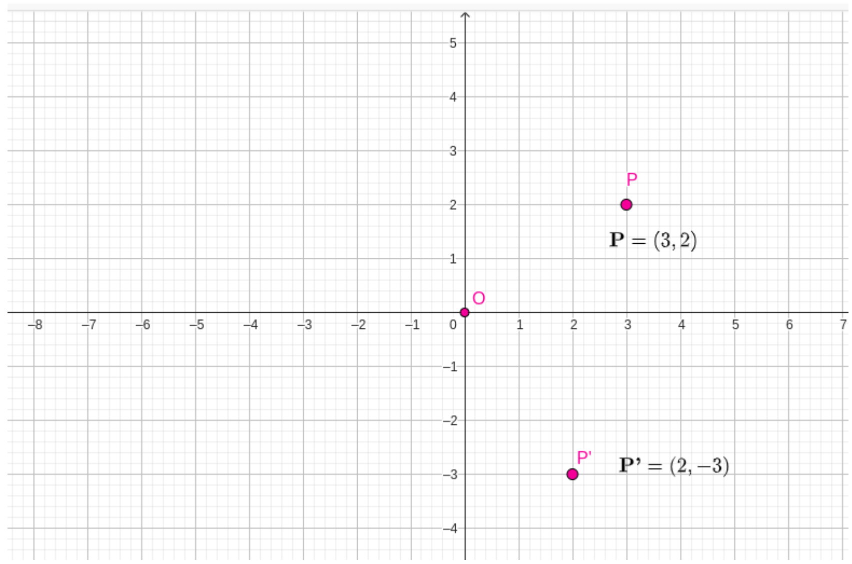


(iv)**+360°**

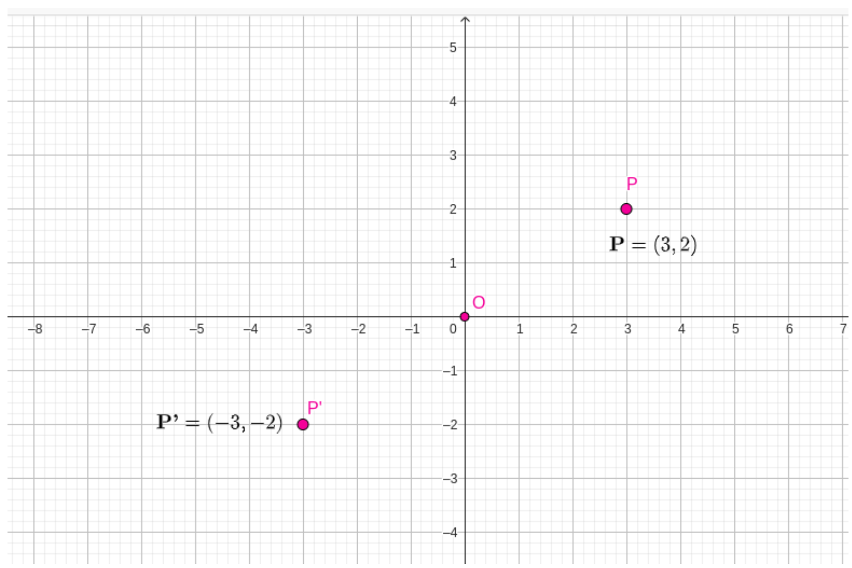


(d) Similary rotate the point in a clockwise direction around the origin with different angles as shown below:

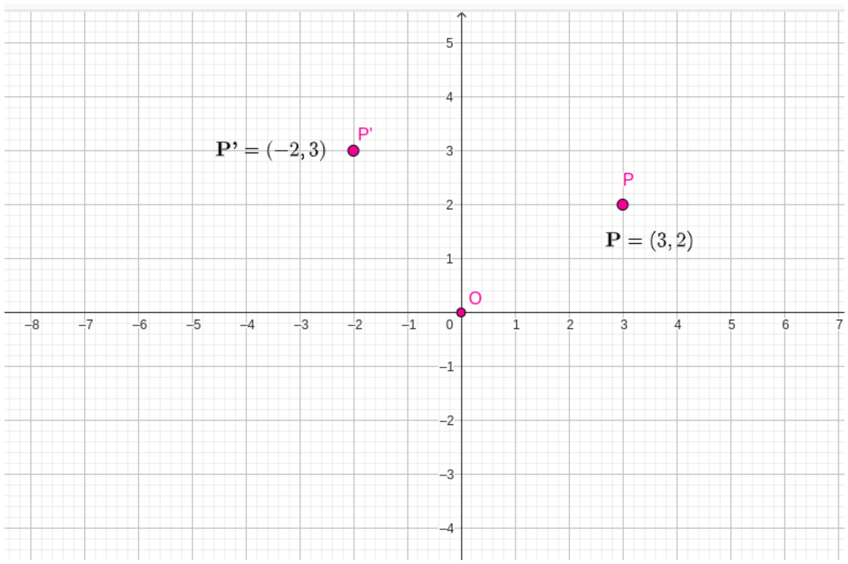
(i)**-90°**



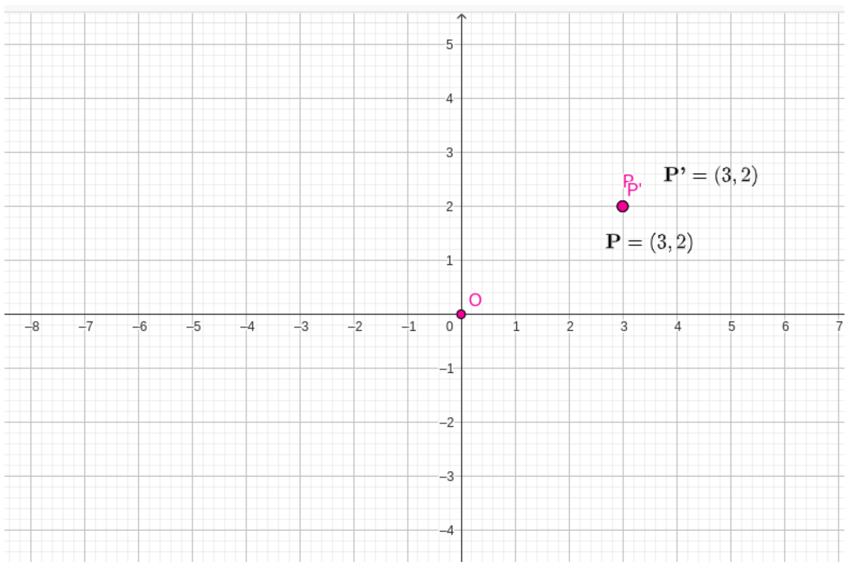
(ii)**=-180°**



(iii)**-270°**



(iv)**-360°**



Key Takeaway

(a.) The image of point P remains the same when rotated through $\pm 180°$ (clockwise or counterclockwise) about the origin.

(b.) A rotation through $\pm 360°$ and $0°$ about the origin does not change the position of the object.

In summary, a point (p, q) which is rotated through the indicated angles about the origin is shown in the table below.

Table 2.3.6

Angle of rotation	$0°$	$+90°$	$-90°$	$180°$	$-180°$	$+270°$	$+360°$	$-360°$
image of (p,q)	(p, q)	$(-q, p)$	$(q, -p)$	$(-p, -q)$	$(-p, -q)$	$(q, -p)$	(p, q)	(p, q)

The figure below shows triangle ABC and its images after rotations about the origin with different angles of rotation (90°, 180°, 270° and 360°).

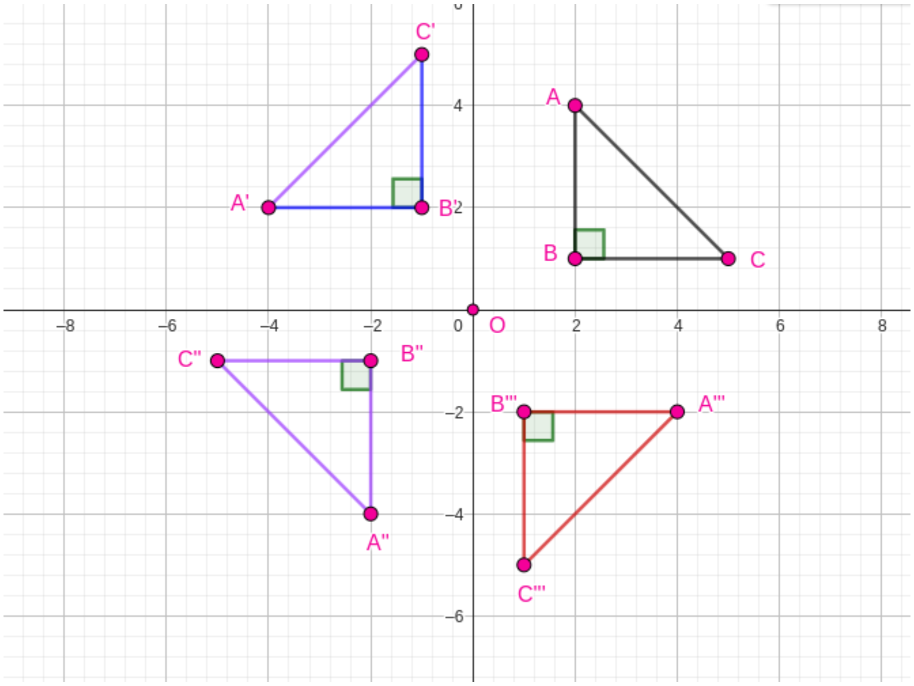


Figure 2.3.7

Table 2.3.8

Object	$A\,(2,4)$ $B\,(2,1)$ $C\,(5,1)$	$A\,(2,4)$ $B\,(2,1)$ $C\,(5,1)$	$A\,(2,4)$ $B\,(2,1)$ $C\,(5,1)$	$A\,(2,4)$ $B\,(2,1)$ $C\,(5,1)$	$A\,(2,4)$ $B\,(2,1)$ $C\,(5,1)$	$A\,(2,4)$ $B\,(2,1)$ $C\,(5,1)$	$A\,(2,4$ $B\,(2,1$ $C\,(5,1$
Angle of rotation	$0°$	$+90°$	$-90°$	$+180°$	$-180°$	$270°$	$+360°$
Image point	$A\,(2,4)$ $B\,(2,1)$ $C\,(5,1)$	$A'\,(-4,2)$ $B'\,(-1,2)$ $C'\,(-1,5)$	$A\,(4,-2)$ $B\,(1,-2)$ $C\,(1,-5)$	$A''\,(-2,-4)$ $B''\,(-2-1)$ $C''\,(-5-1)$	$A''\,(-2,-4)$ $B''\,(-2,-1)$ $C''\,(-5,-1)$	$A'''\,(4,-2)$ $B'''\,(1,-2)$ $C'''\,(1,-5)$	$A\,(2,4$ $B\,(2,1$ $C\,(5,1$

Rotation of Points by a Given Angle Around a Specified Center

Consider a point $A\,(4,3)$ .We are required to finding the coordinates of its image after a Rotation taking the centre to be $(1,2)$ and angle of rotation to be $90°$;

$(4,3)$ is mapped onto $(0,5)$.

Given the point $(4,3)$ and the centre of rotation $(1,2)$, To obtain this point $(0,5)$ without a graph, We follow this steps;

$$\text{x-coordinate} = 1-(3-2) = 0$$
$$\text{y-coodinate} = 2+(4-1) = 5$$
$$\text{Point of the image} = (0,5)$$

from the given points, for point $(4,3)$,We let $p = 4$ and $q = 3$ and for the given centre $(1,2)$ we let $x = 1$ and $y = 2$.

In general a point (p,q) rotated through 90° about the centre (x,y) is mapped on to the poi

Considering the same point P, but now the angle of rotation to be $180°$; To find the coordinates of its image we follow the following steps:

Given the point $(4,3)$ and the centre of rotation $(1,2)$, To find the point of the image we follow the following steps;

$$\begin{aligned}\text{x-coordinate} =&(2 \times 1) - 4\\ =&-2\\ \text{y-coordinate} =&(2 \times 2) - 3\\ =&1\\ \text{Point of the image} =&(-2, 1)\end{aligned}$$

In general a point (p,q) rotated through 180° about the centre (x,y) is mapped on to the po

Example 2.3.9

A triangle ABC with coordinates $A\,(2,1)$, $B\,(3,2)$ and $C\,(3,4)$ is rotated through the centre and angle of $90°$ in a clockwise direction.Find the coordinates of its image

Solution. Since triangle ABC is rotated in a clockwise direction with an angle of $90°$ through the origin, then the angle of rotation is $-90°$

According to the rule, If we have our points (p, q) which will be mapped to $(q, -p)$ if rotated through the centre and angle of rotation is $-90°$

Therefore, we will individually apply the rotation formula to all three given points.

$$\begin{aligned}A(2,1) \to& A'(1,-2)\\ B(3,2) \to& B'(2,-3)\\ C(3,4) \to& C'(4,-3)\end{aligned}$$

The coordinates of triangle $A'B'C'$ are $A'\,(1,-2)$, $B'\,(2,-3)$ and $C'\,(4,-3)$

□

Exercises

1. A point $P\,(4,3)$ maps onto $P'\,(-1,4)$ under a rotation R centre $(1,1)$. Find the angle of rotation.
2. Describe the rotation which maps the rectangle whose verticies are $P\,(2,2)$, $Q\,(6,2)$, $R\,(6,4)$ and $S\,(2,4)$ onto a rectangle whose verticies are $P'\,(2,-2)$, $Q'\,(2,-6)$, $R'\,(4,-6)$ and $S'\,(4,-2)$
3. Give the coordinates of the image of each of the following points when rotated through $180°$ in a clockwise direction about $(2,1)$

   (a) (4,-2) (b) (-2,2) (c) (4,4) (d) (-2,-1) (e) (-3,2)
4. Find the coordinates of the verticies of the image of a triangle whose verticies are $P\,(-4,6)$, $Q\,(-4,2)$ and $R\,(-2,2)$ when rotated about the origin through:

   (a) $-90°$ (b) $-180°$ (c) $270°$
5. The parallelogram whose verticies are $A\,(4,4)$, $B\,(8,4)$, $C\,(2,2)$ and $D\,(6,2)$ is rotated to give an image whose verticies are $A'\,(4,-2)$, $B'\,(4,-6)$, $C'\,(2,0)$ and $D'\,(2,-4)$. Find the centre and angle of rotation.

Checkpoint 2.3.10 This question contains interactive elements.

Checkpoint 2.3.11 This question contains interactive elements.

Checkpoint 2.3.12 This question contains interactive elements.

Checkpoint 2.3.13 A triangle ABC with coordinates $A\,(-6, 5), B\,(9, 4)$ and $C\,(6, 7)$ is rotated through the centre and angle of 90° in a clockwise direction.Find the coordinates of its image.

A' = _____
B' = _____
C' = _____

Answer 1. $[5, 6]$

Answer 2. $[4, -9]$

Answer 3. $[7, -6]$

Solution. Worked solution Since triangle ABC is rotated in a clockwise direction with an angle of 90° through the origin, then the angle of rotation is $-90°$

According to the rule, If we have our points (x, y) which will be mapped to $(y, -x)$ if rotated through the centre and angle of rotation is $-90°$

Therefore, we will individually apply the rotation formula to all three given points.

$A = (-6, 5) \Rightarrow A' = [5, 6]$
$B = (9, 4) \Rightarrow B' = [4, -9]$
$C = (6, 7) \Rightarrow C' = [7, -6]$

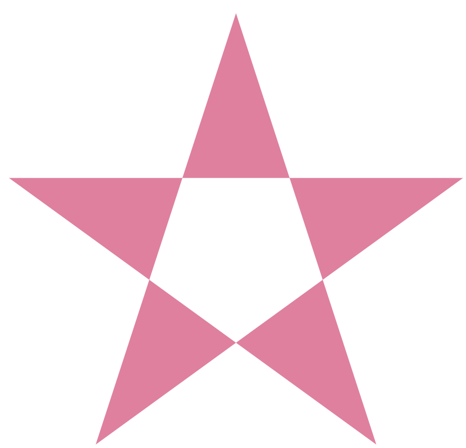
The coordinates of triangle $A'B'C'$ are A' $[5, 6]$, B' $[4, -9]$ and C' $[7, -6]$

2.3.3 Rotational Symmetry

2.3.3.1 Determining the Order of Rotational Symmetry of Plane Figures

Activity 2.3.4 Work in groups
Materials

- A printed copy of the figure.
- Pencils, push pin.
- Constuction paper.
- Pair of scissors.



Instructions

- On a construction paper, trace and cut the figure above.
- Place the tracing on top the printed copy and place a pin through their centre such that the tracing can rotate.

- Manually rotate the tracing around the centre and note how many times the shape looks exactly the same in one full turn (360°).

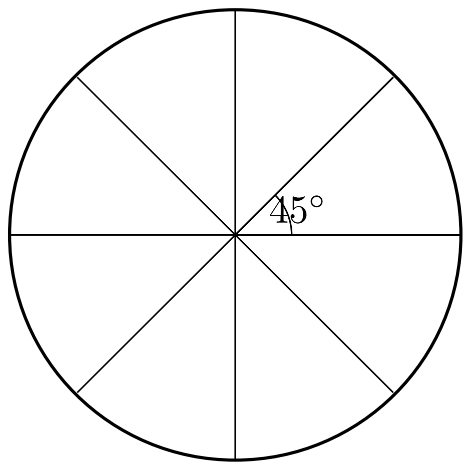
Key Takeaway

The number of times the tracing of the star fits onto the printed copy in one complete turn is 5 times. This is called the **order of rotational symmetry,** that is, the number of times the figure fits onto itself in one complete turn (360°).

When given a figure with the measure of the angle between the identical parts, the order of rotational symmetry can be computed as shown.

$$\text{Order of rotational symmetry} = \frac{360^\circ}{\text{angle between the identical parts}}$$

Example 2.3.14 Find the order of rotational symmetry in the figure below.



Solution.

$$\text{Order of rotational symmetry} = \frac{360^\circ}{\text{angle between the identical parts}}$$

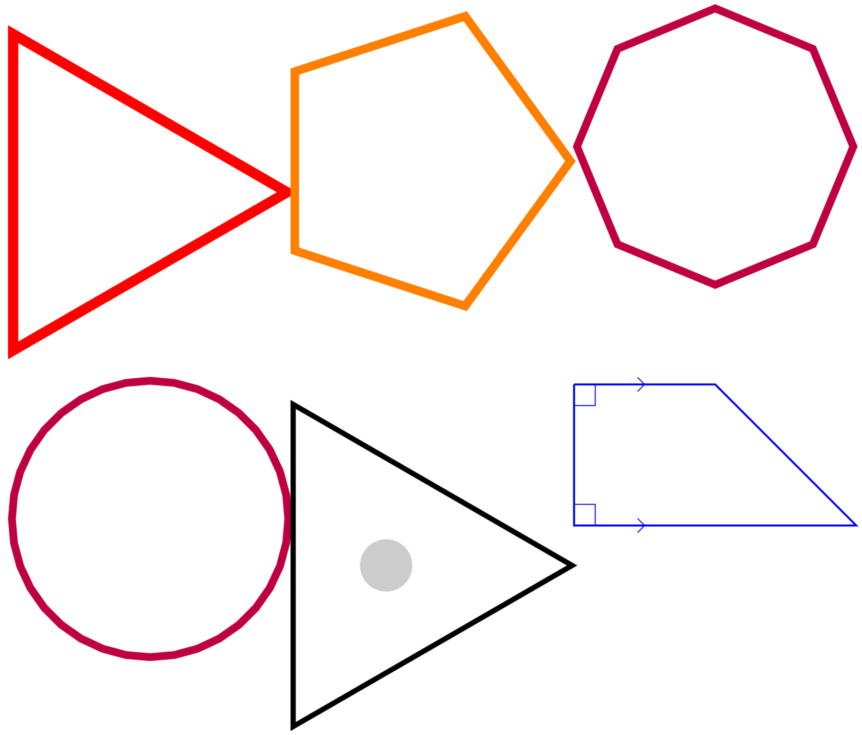
$$\text{Order of rotational symmetry} = \frac{360^\circ}{45^\circ}$$

$$\text{Order of rotational symmetry} = 8$$

□

Exercises

1. State the order of symmetry in the figures below.



2. Find the order of rotational symmetry in the letters of the alphabet.

2.3.3.2 Determining the Axis of Rotation and Order of Rotational Symmetry in Solids

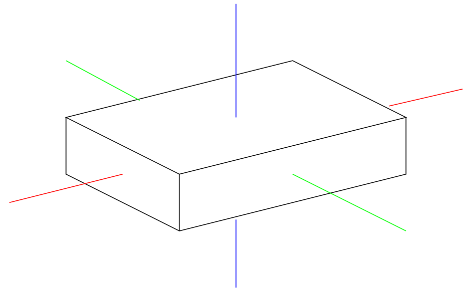
Activity 2.3.5 Here is an activity to explore on axis of rotation of a box (cuboid).

Materials

- A cuboid shaped box
- Three strings
- Pins, ruler and pencil

Instructions

- Measure and note down the cuboid's dimensions (length, width, height).
- Mark the centre of box on each face using a pencil and make holes using pins through the centres.
- Put the strings through the holes such that they appear as shown.
- Suspend the cuboid and spin it around each of the strings and observe the alignment of the cuboid. Does the box appear to be the same as you rotate?



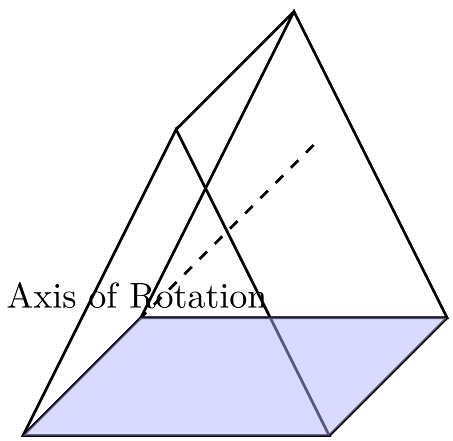
Key Takeaway

- A solid has rotational symmetry if it can be rotated about a fixed straight line and still appears to be the same.
- The straight line around which the object is rotated is called **axis of rotation.** In the activity, the strings represents the axes of rotational symmetry for the cuboid.

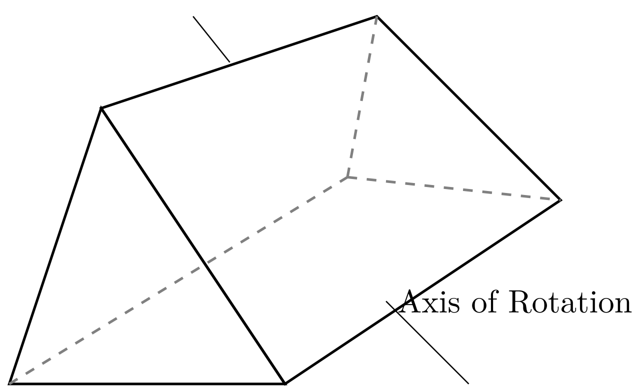
Example 2.3.15 Find the axes of rotation for a triangular pyramid whose cross-section is an equilateral triangle.

Solution. The figure below shows a triangular prism whose cross-section is an equialteral triangle.

The axis of rotation passes through the traingular face. Therefore, the order of rotation through this axis is 3.



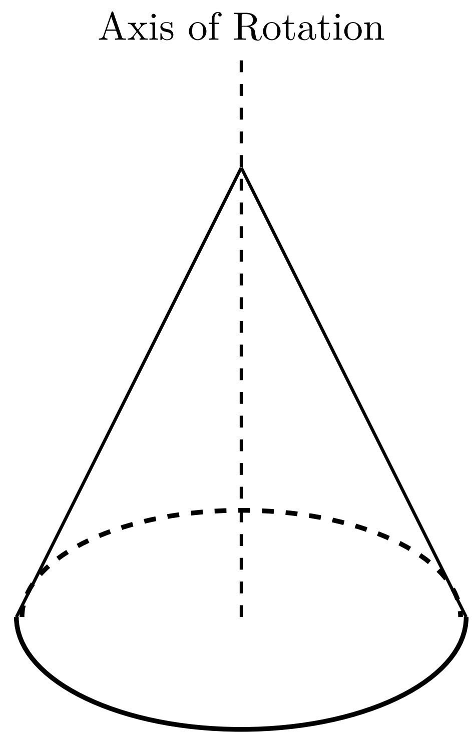
The prism also has other 3 axes of rotation with each axis having 2 orders of rotational symmetry as shown in the figure below:



□

Example 2.3.16 Find the axis of rotation of a cone. What is the order of rotational symmetry?

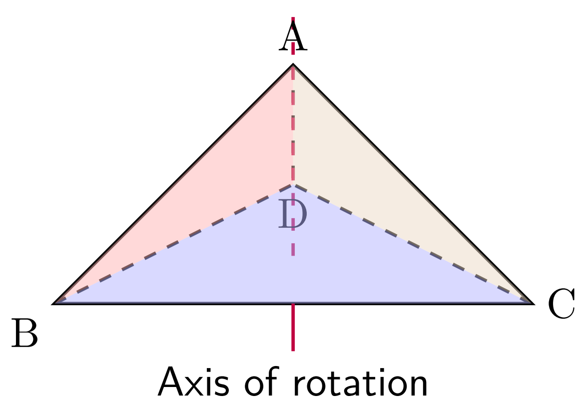
Solution. A cone has one axis of rotation with infinite numbers of order of rotational symmetry since its base is circular.



□

Exercises

1. Find the other axes of rotation and order of rotational symmetry of the regular tetrehedron given one of the axes from A and passing at the center of the face BDC..



2. Find the axes of rotation and order of rotational symmetry of a triangular base pyramid whose base is:

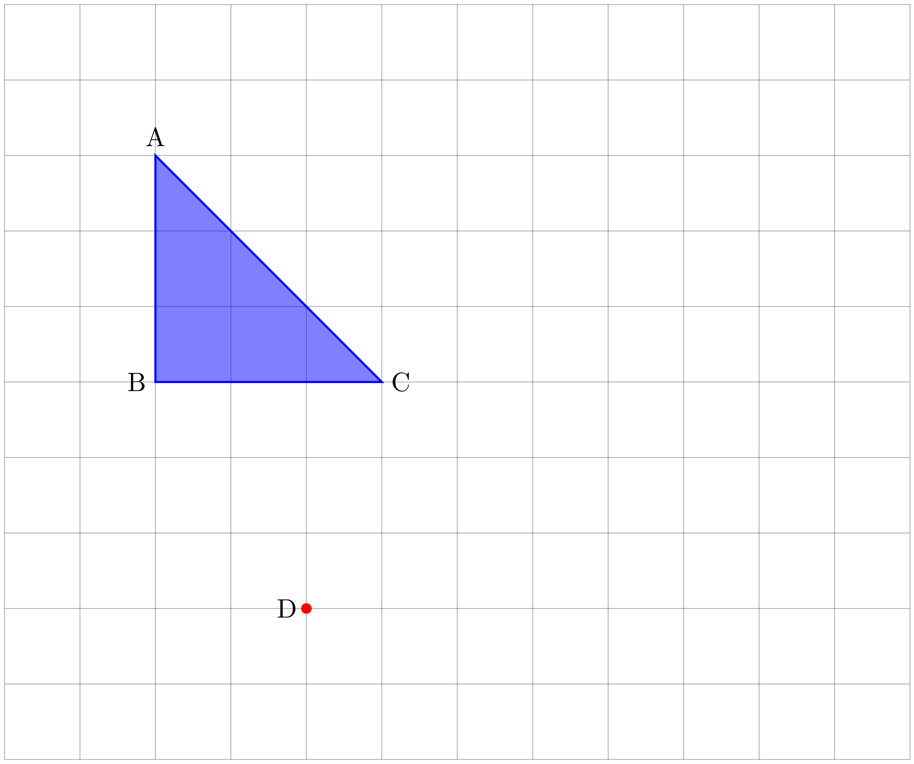
- Scalene triangle
- Isosceles triangle

2.3.4 Rotation and Congruence

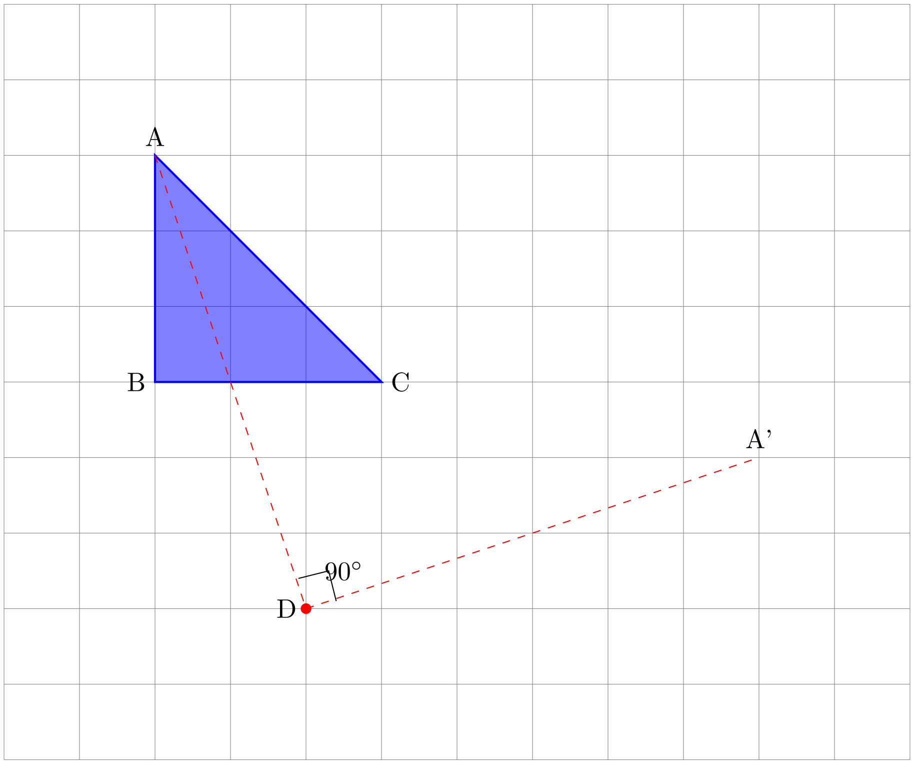
Activity 2.3.6 Work in groups

Copy the trangle ABC and the point D on a graph paper. Using a ruler and a protractor, rotate the triangle $-90°$ about point D.

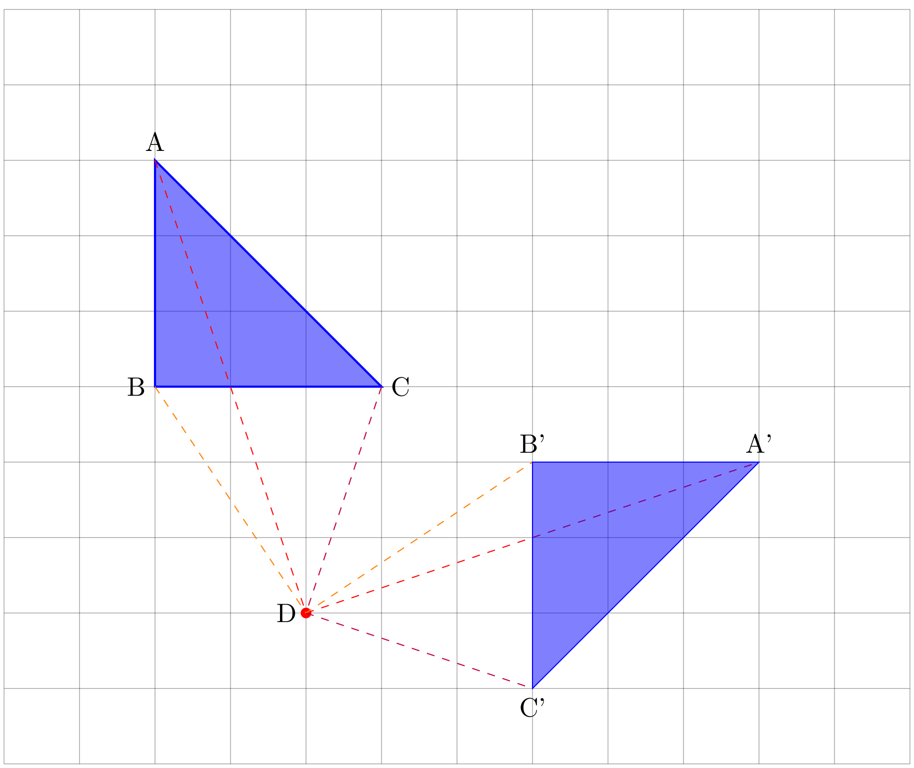
Draw a dotted line to connect vertex A to point D



Place a protractor at the line AD with the centre of the protractor at D and measure $90°$. Using a ruler draw DA' such that $AD = DA'$.



Repeat the step above for for vertices B and$C$.



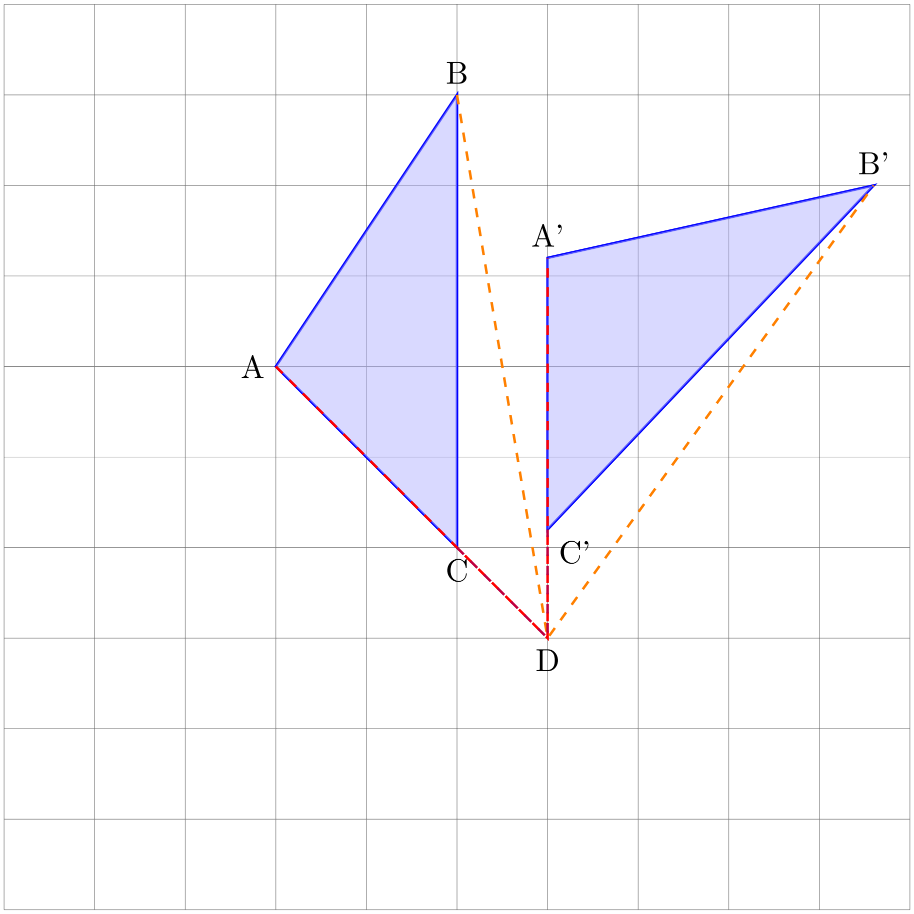
Key Takeaway

Congruence refers to a relationship between two figures or objects, whereby, they are identical in size and shape.

Rotation is a type of transformation that repositions an object but preserves the shape and size of the object. Thus, rotation produces congruent figures.

ΔABC and $\Delta A'B'C'$ are similar in size and shape. Therefore, they are said to be directly congruent.

Example 2.3.17 Triangle ABC is mapped onto $A'B'C'$ after a rotation of $-45°$ and centre of rotation D.



- ΔABC and $\Delta A'B'C'$ have the same shape and size.
- The length of the corresponding sides of ΔABC and $\Delta A'B'C'$ are the same.
- Every corresponding internal angle for the triangles remain the same.

Therefore, ΔABC and $\Delta A'B'C'$ are said to be **directly congruent.** □

Checkpoint 2.3.18 This question contains interactive elements.

Checkpoint 2.3.19 An error occurred while processing this question.

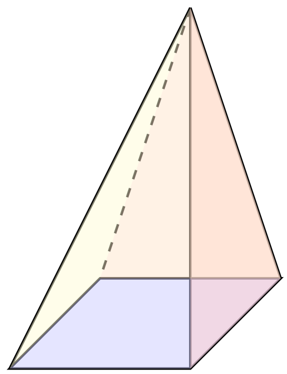
Exercises

1. Identify the axes of rotational symmetry and their respective order in the following:

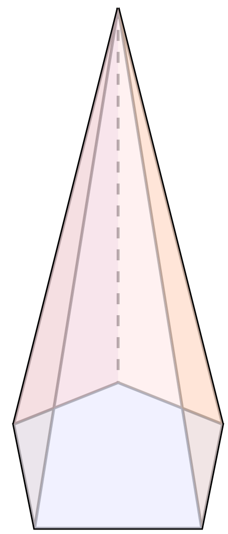
- Cylinder
- Rectangular pyramid

- Sphere
- Cube

2. Identify the axes of rotational symmetry and their respective order for the following figures:



square-base pyramid



pentagonal-base pyramid

2.4 Trigonometry 1

By the end of this topic, you should be able to:

1. Define and apply trigonometric ratios (sine, cosine, tangent).
2. Use the Pythagorean Theorem to find missing sides in right-angled triangles.
3. Solve for unknown sides and angles using trigonometric ratios.
4. Apply trigonometric values of special angles (30°, 45°, 60°) in solving geometric problems.
5. Analyze and interpret relationships between angles and side lengths in various contexts to solve application problems.

2.4.1 Trigonometric Ratios of Acute Angles

2.4.1.1 Tangent of an Acute Angle

Activity 2.4.1 Work in pairs
What you require:

- A piece of paper
- A ruler
- A pencil

1. Using a piece of paper, a ruler, and a pencil, carefully draw the diagram shown below.

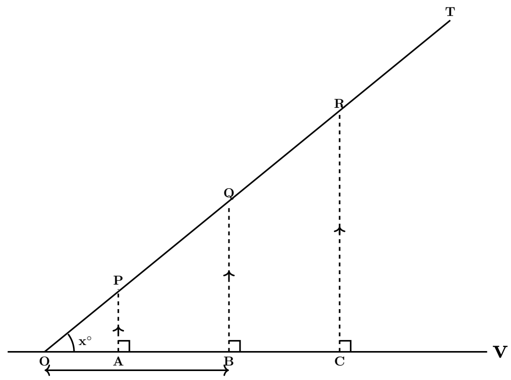


Figure 2.4.1 fig 1.2

2. Measure and record the lengths of OB, BQ, OC, CR, OA, and APusing a ruler..

3. Identify whether triangles OPA, OQB, and ORC are similar.

   If they are similar, compare the ratios of their corresponding sides.

4. Calculate the following ratios:

   $\frac{PA}{OA}$

   $\frac{QB}{OB}$

   $\frac{RC}{OC}$

5. Observe the ratios from the previous step.

   What do you notice about the relationship between them?

6. Considering the parallel lines BQ, AP and RC examine the relationship between the vertical and horizontal distances.

   What do you notice about their ratios?

7. Use a protractor to measure the angle marked $x°$ in the diagram.

8. Share your observations and conclusions with your classmates.

Extended Activity

1. The inclination of the observer's line of sight to the top of a $10\,m$ high flag pole, positioned $15\,m$ away, can be determined using a scale drawing, as illustrated in the diagram below.

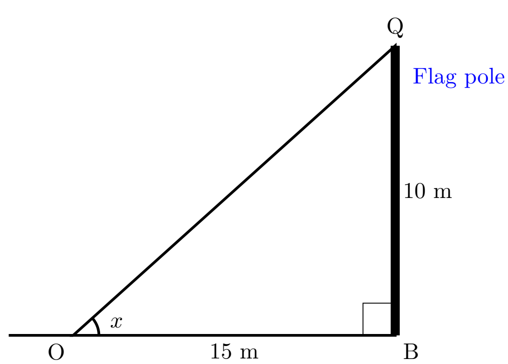


Figure 2.4.2

What do we call the angle represented by x?

2. Look at the triangle shown in **Figure 2.4.3.**

   Express $tan\,\alpha$ in terms of the lengths of the sides of the triangle.

   Hint: Recall that in a right-angled triangle:

$$tan\,\alpha = \frac{\text{opposite side}}{\text{adjacent side}}$$

Key Takeaway
When given the triangle below, you will notice the following:

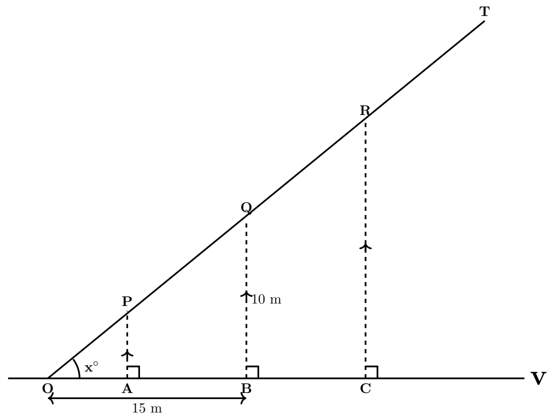


Figure 2.4.3

(a) The triangles OPA, OQB, and ORC are similar.

This means that the ratios of their corresponding sides are equal:

$$\frac{PA}{OA} = \frac{QB}{OB} = \frac{RC}{OC} = \frac{15}{10} = 1.5$$

(b) For any line parallel to BQ, rthe ratio of vertical distance to horizontal distance remains the same in each triangle. In this case, the ratio is1.5.

This constant ratio, $\frac{\text{Vertical distance}}{\text{Horizontal distance}}$ is called the **tangent of angle** VOT.

Therefore, the tangent of $x°$ is 1.5 which can be written as:

$$\tan x° = 1.5$$

The tangent of an angle depends only on the size of the angle, not on the triangle's size.

The diagram below shows a right-angled triangle ABC, where:

$$\angle ABC = \theta.$$

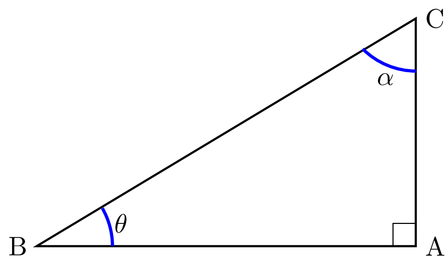


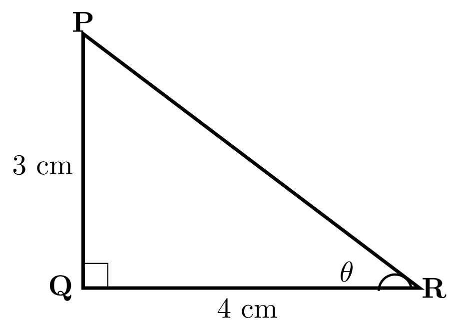
Figure 2.4.4

- The side AC is the vertical side, which is **opposite** to angle θ.
- The side AB is the horizontal side, which is **adjacent** to angle θ
- The side BC is the **hypotenuse**, which is the longest side of the right-angled triangle.

In this case, the tangent of angle θ is defined as the ratio of the opposite side to the adjacent side:

$$\tan\theta = \frac{\textbf{Opposite side}}{\textbf{Adjacent side}} = \frac{AC}{AB}.$$

Example 2.4.5 Find the tangent of the indicated angles using the given measurements below.

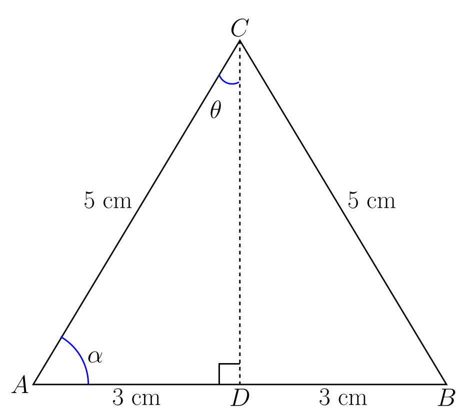


Solution. The above diagram the sides given are,
Opposite side $= 3\,cm$
Adjacent side $= 4\,cm$
Therefore,

$$\begin{aligned}\tan\theta =& \frac{\text{Opposite}}{\text{Adjacent}}\\ =& \frac{3\,cm}{4\,cm}\\ =& \frac{3}{4}\\ =& 0.75\end{aligned}$$

Therefore, $\tan\theta = 0.75$ □

Example 2.4.6 Find the tangent in the indicated angle below.



Solution. The first thing you should do is to calculate the perpendicular height of the Triangle then identify the opposite and the adjacent sides of angle θ and angle α. Finaly find there tangents.

Finding the perpendicular height.

We use pythagorean relationship that is

$$H^2 = b^2 + h^2$$

$$\begin{aligned} h^2 &= H^2 - b^2 \\ &= 5^2 = 3^2 \\ &= 25 - 9 \\ &= 16 \\ \sqrt{h^2} &= \sqrt{16} \\ h &= 4\,cm \end{aligned}$$

Finding tanθ
$\tan\theta = \frac{\text{Opposite}}{\text{Adjacent}}$
The opposite side $= 3\,cm$
The adjacent side $= 4\,cm$

$$\begin{aligned} \tan\theta &= \frac{3\,cm}{4\,cm} \\ &= \frac{3}{4} \\ &= 0.75 \end{aligned}$$

Finding tanα
$\tan\alpha = \frac{\text{Opposite}}{\text{Adjacent}}$
The opposite side $= 4\,cm$
The adjacent side $= 3\,cm$

$$\begin{aligned} \tan\theta &= \frac{4\,cm}{3\,cm} \\ &= \frac{4}{3} \\ &= 1.3333333333 \\ &= 1\frac{1}{3} \end{aligned}$$

Therefore,

$$\begin{aligned} \tan\theta &= 0.75 \\ \tan\alpha &= 1.333 \end{aligned}$$

□

Tables of Tangents

Activity 2.4.2 Work in groups

What you require: Printed Table of Tangent, a $30\,cm$ ruler, pencil, and calculator (for verification).

1. What is the tangent of an angle?
2. How do we use a Table of Tangents?
3. Use your Table of Tangent to find the folowing tangents.

   a $42°$

   b $35°$

   c $90°$

   d $42° 47$

4. Discuss your findings with other groups in your class.

Key Takeaway

Special tables have been prepared and can be used to obtain tangents of acute angle (see tables of natural tangents in your mathematical tables). The technique of reading tables of tangents is similar to that of reading tables of logarithms or square roots.

Here's how you can use it:

(i) Identify the angle

- Find the given angle in the leftmost column (if degrees) or the top row (if radians).

(ii) Locate the Tangent Value

- Read across the row (or down the column) to find the corresponding tangent value.

Note

1. In the tables of tangents, the angles are expressed in decimals and degrees or in degrees and minutes.
2. One degree is equal to 60 (60 minutes) . Thus, $30 = 0.50°$, $54 = 0.9°$ and $6 = 0.1°$..

From the table, the values of tangents increase as the angles approach $90°$

Example 2.4.7 Find the tangent of each of the following angles from the table:

(a) $60°$

(b) $52°$

(c) $46.7°$

(d) $52° 47$

Solution.

(a) Find $tan(60°)$

Locate $60°$in the table.

Read the corresponding tangent value. that is

$$tan(60°) = 1.732$$

(b) Find $tan(52°)$

Locate 52° in the table.

Read the corresponding tangent value. that is

$$tan(52°) = 1.279$$

(c) Find $tan(46.7°)$

Locate 46.7° in the table.

Read the corresponding tangent value. that is

$$tan(46.7°) = 1.0612$$

(d) Find $tan(52° 47)$

Convert 47 to degrees by deviding by 60

$$(\frac{47}{60})° = 0.78°$$

Therefore,

$$tan\, 52° 47 = 52.78°$$

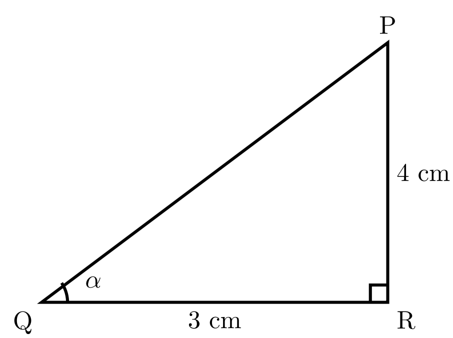
Using decimal tables, $tan\, 52.7° = 1.3127$. From the difference column under 8 reads 0.0038

Therefore,

$$\begin{aligned} tan\, 52° 47 &= 52.78° \\ &= 1.3127 + 0.0038 \\ &= 1.3165 \end{aligned}$$

□

Example 2.4.8 Using natural tangents in your mathematical tables, find α as shown in the figure below.



Solution. Opposite $= 4\,cm$
Adjacent $= 3\,cm$

$$\begin{aligned} Tan\,\alpha &= \frac{\text{Opposite}}{\text{Adjacent}} \\ &= \frac{4\,cm}{3\,cm} \end{aligned}$$

$$=1.3333$$

Note that 1.3333 cannot be read directly from the tables of tangents. Therefore, look for a number nearest to 1.3333 from the tables. In this case, the nearest number is 1.3319 The angle whose tangent is 1.3319 is 53.1°

The difference between 1.3333 and 1.3319 is 14.. From the difference column in the tangent tables, the nearest number to 14. is 9 which gives a difference of 0.44.

Adding 0.44 to 53.1° we get 53.54°.

Therefore, the angle whose tangent is 1.3333 = 53.54°

Thus, $\alpha = 53.54°$. □

Checkpoint 2.4.9 Exact Values of Sine, Cosine, and Tangent Without a Calculator. Load the question by clicking the button below.Without using a calculator write down the following values

1. $\sin 120° =$ ______
2. $\sin 150° =$ ______
3. $\cos 120° =$ ______
4. $\cos 150° =$ ______
5. $\tan 120° =$ ______
6. $\tan 150° =$ ______

Answer 1. $\frac{\sqrt{3}}{2}$

Answer 2. $\frac{1}{2}$

Answer 3. $-\frac{1}{2}$

Answer 4. $-\frac{\sqrt{3}}{2}$

Answer 5. $-\sqrt{3}$

Answer 6. $-\frac{1}{\sqrt{3}}$

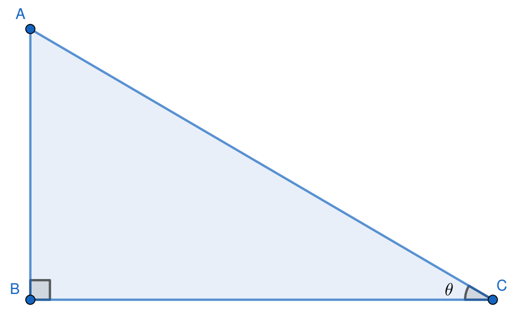
Solution.

1. $\sin 120° = \sin(180 - 60) = \sin 60° = \dfrac{\sqrt{3}}{2}$
2. $\sin 150° = \sin(180 - 30) = \sin 30° = \dfrac{1}{2}$
3. $\cos 120° = -\cos 60 = -\dfrac{1}{2}$
4. $\cos 150° = -\cos 30° = -\dfrac{\sqrt{3}}{2}$
5. $\tan 120° = \dfrac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = -\sqrt{3}$
6. $\tan 150° = \dfrac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = -\dfrac{1}{\sqrt{3}}$

Checkpoint 2.4.10 Finding the Tangent of a Right Angle Triangle. Load the question by clicking the button below.

This question contains interactive elements.

Checkpoint 2.4.11 Finding Trigonometric Ratios in a Right Angled Triangle (1). Load the question by clicking the button below.Find the values of the $\sin(\theta)$, $\cos(\theta)$, and $\tan(\theta)$ in the given right-angled triangle if AB is of length 1 and BC is of length 4. Enter exact values, using e.g. `sqrt(2)` for $\sqrt{2}$.



$\sin(\theta) =$ _____; $\cos(\theta) =$ _____; $\tan(\theta) =$_____.

Answer 1. $\frac{1}{\sqrt{17}}$

Answer 2. $\frac{4}{\sqrt{17}}$

Answer 3. $\frac{1}{4}$

Solution. By Pythagoras the length of the hypotenuse AC is

$$\sqrt{AB^2 + BC^2} = \sqrt{1^2 + 4^2} = \sqrt{17}.$$

The trignometric values are then\begin{align*}\sin(\theta) &= \frac{AB}{AC} = \frac{{1}}{{\sqrt{17}}}\\ \cos(\theta) &= \frac{BC}{AC} = \frac{{4}}{{\sqrt{17}}}, \ \text{and}\\ \tan(\theta) &= \frac{AB}{BC} = \frac{{1}}{{4}}.\end{align*}

Exercises

1. Read from tables the tangent of:

   a 88° 46

   b 60° 46

   c 45°

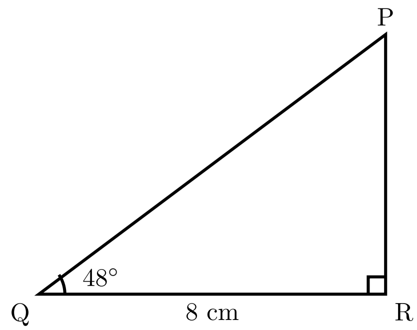
2. Express each of the following in degrees and minutes:

   a 26.75°

   b $40\frac{1}{2}^\circ$

   c $56\frac{1}{4}^\circ$

3. Use Natural logarithm of tangents to find the length PR in the figure below. (leave your answer to 2 **decimal places**)



4. A ladder leans against a wall so that its foot is $4.5\,m$ away from the foot of the wall and its top is 10 up the wall. Calculate the angle it makes with the ground .

5. In a right-angled triangle, the shorter sides are $6.5\,cm$ and $12.2\,cm$ long. Find the sizes of its acute angles.

2.4.1.2 Sine and Cosine of an Acute Angle

Activity 2.4.3 Work in groups

What you require; A piece of paper, a ruler and a pencil.

1. The figure below shows AP, BQ, **and** CR perpendicular to OV and $\angle TOV = \theta$

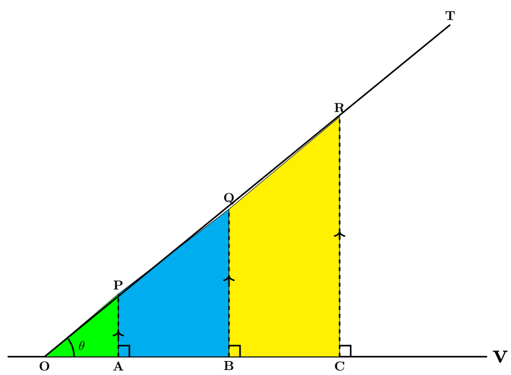


Figure 2.4.12 fig 1.5

2. Copy the above figure in your writing materials.

   Measure lengths OA, OP, AP, OQ, OB, BQ, OR, OC and CR

3. Fill in the following;

   i $\frac{AP}{OP}$ =______

   ii $\frac{BQ}{OQ}$ =______

   iii $\frac{CR}{OR}$ =______

4. What do you notice about the ratios of roman (i...iii).

5. Fill also the following;

   i $\frac{OA}{OP}$ =______

   ii $\frac{OB}{OQ}$ =______

   iii $\frac{OC}{OR}$ =______

6. What do you notice about these ratios (5) above.

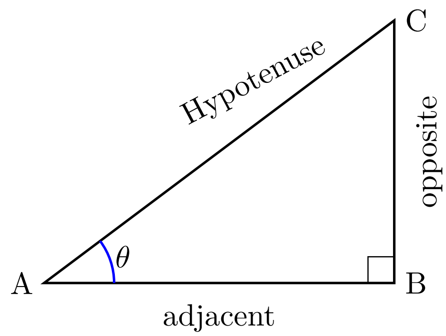
7. Discuss your findings with other groups in your class.

Key Takeaway
You will notice that,

a) The ratios of (3) are the same and is expressed as;
$\frac{AP}{OP} = \frac{BQ}{OQ} = \frac{CR}{OR}$
This constant value is obtained by taking the ratio of the side **opposite** to the angle θ to the **hypotenuse** side in each case. This ratio is called the sine of angle θ, which can be written as as $sin\,\theta$.

b) The ratios of (5) are the same and is expressed as;
$\frac{OA}{OP} = \frac{OB}{OQ} = \frac{OC}{OR}$
This constant value is obtained by taking the ratio of the side **adjacent** to the angle θ to the **hypotenuse** side in each case. This ratio is called the cosine of angle θ, which can be written as $cos\,\theta$.

In general, given a right-angled triangle whith **opposite side, adjacent side** and **hypotenuse side** as shown,



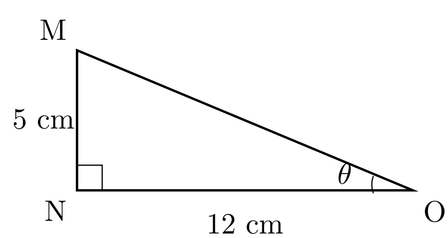
$$tan\,\theta = \frac{\text{Opposite}}{\text{Adjacent}}$$
$$cos\,\theta = \frac{\text{Adjacent}}{\text{Hypotenuse}}$$
$$sin\,\theta = \frac{\text{Opposite}}{\text{Hypotenuse}}$$

The above formula also applies to the trigonometric ratios for α.

Example 2.4.13 In the figure below, $MN = 5\,cm$, $NO = 12\,cm$ and $\angle MNO = 90°$. Calculate:

(a) $sin\,\theta$

(b) $cos\,\theta$



Solution.

(a)

$$\begin{aligned} sin\,\theta =& \frac{\text{opposite}}{\text{hypotenuse}} \\ =& \frac{MN}{MO} \\ =& \frac{5}{MO} \end{aligned}$$

Recall:

$$\begin{aligned} MO^2 =& 12^2 + 5^2 \\ =& 144 + 25 \\ =& 169 \\ MO =& 13\,cm \end{aligned}$$

Thus,

$$\begin{aligned} sin\,\theta =& \frac{5}{13} \\ =& 0.3846 \end{aligned}$$

(b)

$$\begin{aligned} cos\,\theta =& \frac{\text{adjacent}}{\text{hypotenuse}} \\ =& \frac{NO}{MO} \\ =& \frac{12}{13} \\ =& 0.9231 \end{aligned}$$

□

Example 2.4.14 A ladder leans against a wall, forming a 70° angle with the ground. If the ladder is 5 meters long, how high does it reach on the wall?

Solution. Using sin , since we need the opposite side:

$$\begin{aligned} sin\,70^\circ =& \frac{\text{height}}{\text{hypotenuse}} \\ 0.9397 =& \frac{\text{height}}{5} \\ \text{height} =& 5 \times 0.9397 \\ =& 4.6985\,m \end{aligned}$$

The ladder reaches $4.6985\,m$ up the wall. □

Checkpoint 2.4.15 Exact Values of Sine and Cosine Without a Calculator. Load the question by clicking the button below.Without using a calculator write down the following values

1. $\sin 120^\circ =$ ______

2. $\sin 150° =$ ______
3. $\cos 120° =$ ______
4. $\cos 150° =$ ______
5. $\tan 120° =$ ______
6. $\tan 150° =$ ______

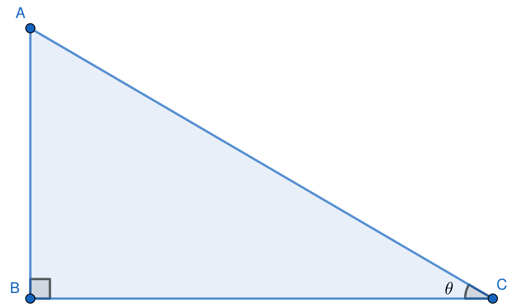
Answer 1. $\frac{\sqrt{3}}{2}$
Answer 2. $\frac{1}{2}$
Answer 3. $-\frac{1}{2}$
Answer 4. $-\frac{\sqrt{3}}{2}$
Answer 5. $-\sqrt{3}$
Answer 6. $-\frac{1}{\sqrt{3}}$
Solution.

1. $\sin 120° = \sin(180 - 60) = \sin 60° = \dfrac{\sqrt{3}}{2}$
2. $\sin 150° = \sin(180 - 30) = \sin 30° = \dfrac{1}{2}$
3. $\cos 120° = -\cos 60 = -\dfrac{1}{2}$
4. $\cos 150° = -\cos 30° = -\dfrac{\sqrt{3}}{2}$
5. $\tan 120° = \dfrac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = -\sqrt{3}$
6. $\tan 150° = \dfrac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = -\dfrac{1}{\sqrt{3}}$

Checkpoint 2.4.16 Finding Ladder Height and Base Distance. Load the question by clicking the button below.

This question contains interactive elements.

Checkpoint 2.4.17 Finding Trigonometric Ratios in a Right Angled Triangle (2). Load the question by clicking the button below.Find the values of the $\sin(\theta)$, $\cos(\theta)$, and $\tan(\theta)$ in the given right-angled triangle if AB is of length 1 and BC is of length 4. Enter exact values, using e.g. `sqrt(2)` for $\sqrt{2}$.



$\sin(\theta) =$ _____; $\cos(\theta) =$ _____; $\tan(\theta) =$_____.

Answer 1. $\frac{1}{\sqrt{17}}$

Answer 2. $\frac{4}{\sqrt{17}}$

Answer 3. $\frac{1}{4}$

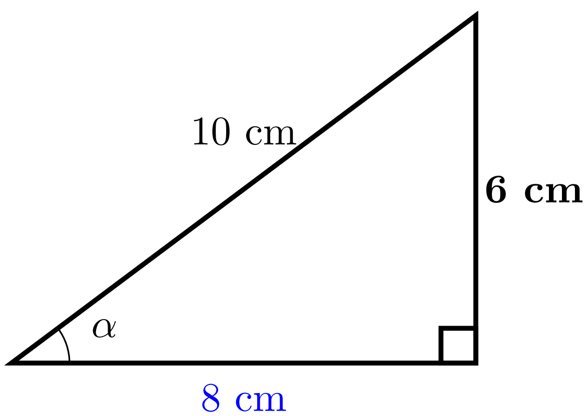
Solution. By Pythagoras the length of the hypotenuse AC is

$$\sqrt{AB^2 + BC^2} = \sqrt{1^2 + 4^2} = \sqrt{17}.$$

The trignometric values are then\begin{align∗}\sin(\theta) &= \frac{AB}{AC} = \frac{{1}}{{\sqrt{17}}}\\ \cos(\theta) &= \frac{BC}{AC} = \frac{{4}}{{\sqrt{17}}}, \ \text{and}\\ \tan(\theta) &= \frac{AB}{BC} = \frac{{1}}{{4}}.\end{align∗}

Exercises.

1. In the figure given below,



   Find;

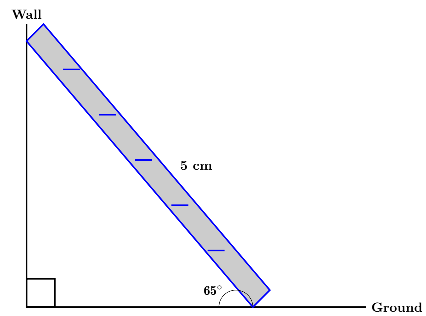
   a) $sin\,\alpha$
   b) $cos\,\alpha$

2. A flagpole 12 meters tall casts a shadow of 8 meters on the ground.

   (a) What is the angle of elevation of the sun?
   (b) If the shadow increases to 10 meters, what will be the new angle of elevation?

3. An airplane takes off at an angle of 18° to the ground. After flying 500 meters,
   (a) How high is the airplane above the ground?
   (b) How far has it traveled horizontally from the starting point?

4. A ladder 5 meters long leans against a vertical wall, making an angle of 65° with the ground as shown.



   (a) How high does the ladder reach on the wall?
   (b) How far is the base of the ladder from the wall?

Tables of Sines and Cosines

Activity 2.4.4 Work in groups

What you require: A scientific calculator{for verification}, ruler and pencil, a printed table of sine and cosine values.

1. Complete the Trigonometric Table.

   Fill in the missing values in the table below. Use a calculator to check your answers if needed.

Angle(°)	$sin\,\theta$	$cos\,\theta$	$sin(90° - \theta)$	$cos(90° - \theta)$
0°	0.0000	1.0000		
30°	0.5000	1.0000		
45°				
60°				
45°	1.0000	0.0000		

2. Analyze the following.
   - Compare $sin\,\theta$ with $cos(90° - \theta)$. What do you notice?
   - Compare $sin\,\theta$ with $cos(90° - \theta)$. What pattern do you see?

3. Discuss your work with other groups in class.

Extended Activity

Activity 2.4.5 Work in groups

What you require: Ruler, pencil, graph paper and a protractor.

1. Draw a right triangle and label the opposite, adjacent and hypotenuse on all three sides and indicate the angle θ.

2. Measure both the length(cm) and angle θ.

3. Find $sin(\theta)$, $cos(\theta)$ and $tan\,\theta$.

4. Divide $sin(\theta)$ by $cos(\theta)$ and record your answers on the table below.

Side Measurement	Length (cm)	Ratio calculation	Results
Opposite		$sin(\theta) = \frac{\text{Opposite}}{\text{Hypotenuse}}$	
Adjacent		$cos(\theta) = \frac{\text{Adjacent}}{\text{Hypotenuse}}$	
Hypotenuse		$tan(\theta) = \frac{\text{Opposite}}{\text{Adjacent}}$	

5. Repeat the procedure with a different right triangle.

6. How are $sin\,\theta$, $cos\,\theta$, and $tan\,\theta$ related?

7. Discuss your work with other learners.

Key Takeaway
When you look at the tables of Cosine and sine, you will notice that,

i) The values of their sines increase from 0 to 1.

ii) The values of their cosines decrease from 1 to 0

Therefore, the values in the difference column of **cosine** tables have to be subtracted and those in the difference columns of the **sine** tables have to be added.

Generally,
The values of sine and cosine ranges from 0 to $1 (0 \leq \theta \leq 1)$.
As the angles increase from $0°$ to $90°$:

Example 2.4.18 Read the sine and cosine values of the following angles from the tables.

(a) $46°$

(b) $45.5°$

(c) $75.67°$

Solution.

(a) $sin\,46°$

Locate 46°in the sine and cosine table.

Read the corresponding sine value. That is

$$sin\,46° = 0.7193,$$

$cos\,46°$

Read the corresponding cosine value. That is

$$cos\,46° = 0.6947,$$

(b) $sin\,45.5°$

Locate 45.5°in the sine and cosine table.

Read the corresponding sine value. That is

$$sin\,45.5° = 0.7133,$$

$cos\,45.5°$

Read the corresponding cosine value. That is

$$cos\,45.5° = 0.7009,$$

(c) $sin\, 75.67°$

Using decimal tables, $sin\, 75.6° = 0.9686$. From the difference column under 7 reads 0.0003

Therefore,

$$\begin{aligned} sin\, 75.67° &= 0.9686 + 0.0003 \\ &= 0.9689 \end{aligned}$$

$cos\, 75.67°$

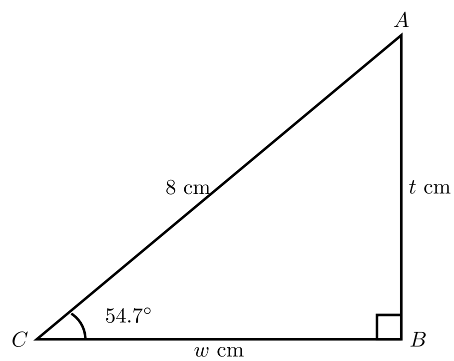
Using decimal tables, $cos\, 75.6° = 0.2487$. From the difference column under 7 reads 0.0012

Therefore,

$$\begin{aligned} cos\, 75.67° &= 0.2487 - 0.0012 \\ &= 0.2475 \end{aligned}$$

□

Example 2.4.19 Find the value of t and w in the figure shown below (Using sine and cosine tables).



Solution. Recall that;
$sin\, \theta = \frac{\text{opposite}}{\text{Hypotenuse}} = \frac{\text{t}}{8}$
$cos\, \theta = \frac{\text{opposite}}{\text{Hypotenuse}} = \frac{\text{w}}{8}$
Therefore,

$$\begin{aligned} sin\, 54.7° &= \frac{\text{t}}{8} \\ 8\, sin\, 54.7° &= \text{t} \end{aligned}$$

But, from tables of sine, $sin\, 54.7° = 0.8161$
Therefore,

$$\begin{aligned} \text{t} &= 8 \times 0.8161 \\ &= 6.5288 \end{aligned}$$

$$\begin{aligned} cos\, 54.7° &= \frac{\text{w}}{8} \\ 8\, cos\, 54.7° &= \text{w} \end{aligned}$$

But, from tables of cosine, $cos\, 54.7° = 0.5779$

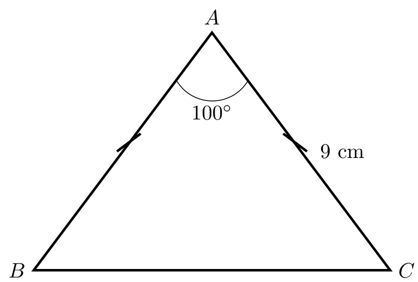
Therefore,

$$\begin{aligned} w =&8 \times 0.5779 \\ =&4.6232 \end{aligned}$$

□

Exercises

1. Find from tables the angle whose sine and cosine is:
   (a) 0.4467
   (b) 0.5875
   (c) 0.0004
2. Read from the tables the sine and the cosine of:
   (a) 45.46°
   (b) 52° 9
   (c) 25° 45
3. The figure below shows an isosceles triangle in which $AB = AC = 9\,cm$. Angle BAC is 100°. Calculate the length of BC.



2.4.2 Sines and Cosines of Complimentary Angles

Activity 2.4.6 Work in pairs

What you require: A pencil, a ruler, a sciantific calculator (**for verification**), working material and a printed sine and cosine table.

1. Read from the tables the values of the following pairs of angles.
   a) $sin\,40°$ $cos\,50°$
   b) $cos\,30°$ $sin\,60°$
   c) $sin\,70°$ $cos\,20°$
   d) $sin\,80°$ $cos\,90°$
2. What do you notice on the results obtained in "1"?
3. Discuss your work with other learners in your class.

Complementary angles are two angles whose sum is 90° (or $\frac{\pi}{2}$radians). The sine and cosine functions of complementary angles have a special relationship:

$$sin(90° - \theta) = cos\,\theta$$

$$cos(90^\circ - \theta) = sin\,\theta$$

For example, if y is one angle and x is the other angle, then:

$$y + x = 90^\circ$$

Generally,For any two complementary angles x and y, the following relationships hold:
$sin\,x = cos\,y$ and $cos\,x = sin\,y$.

This means that the sine of one angle is equal to the cosine of the other and vice versa.

Example 2.4.20 Find acute angles θ and β if:

(a) $cos\,45^\circ = sin\,\alpha$.

(b) $cos\,\beta = sin\,5\beta$.

(c) $sin\,2\alpha = cos\,30^\circ$.

Solution.

(a) $cos\,45^\circ = sin\,\alpha$.
Implies that,

$$\begin{aligned} 45^\circ + \alpha &= 90^\circ \\ &= 90^\circ - 45^\circ \\ &= 45^\circ \end{aligned}$$

(b) $cos\,\beta = sin\,5\beta$.
Implies that,

$$\begin{aligned} \beta + 5\beta &= 90^\circ \\ 6\beta &= 90^\circ \\ \beta &= \frac{90^\circ}{6} \\ &= 15^\circ \end{aligned}$$

(c) $sin\,2\alpha = cos\,30^\circ$.
Implies that,

$$\begin{aligned} 2\alpha + 30^\circ &= 90^\circ \\ 2\alpha &= 90^\circ - 30^\circ \\ \alpha &= \frac{60^\circ}{2} \\ &= 30^\circ \end{aligned}$$

□

Example 2.4.21 A and B are complementary angles.If $A = \frac{1}{2}B$, find:

(A) $sin\,A$

(B) $cos\,A$

Solution. When we say A and B are complimentary it implies that, $A+B=90°$

Also we have, $A=\frac{1}{2}B$

First, Write A interms of A that is,

$$\begin{aligned} A =& \frac{1}{2}B \\ B =& 2A \end{aligned}$$

Since, $A+B=90°$, subtitute $B=2A$

$$\begin{aligned} A+2A =& 90° \\ 3A =& 90° \\ A =& \frac{90°}{3} \\ =& 30° \end{aligned}$$

Therefore,

(a)

$$sin\, A = sin\, 30°$$

Use tables of sine to find $sin\, 30°$

$sin\, 30° = 0.5000$

Therefore, $sin\, A = 0.5000$

(b)

$$cos\, A = cos\, 30°$$

Use tables of cosine to find $cos\, 30°$

$cos\, 30° = 0.8660$

Therefore, $cos\, A = 0.8660$

□

Checkpoint 2.4.22 Finding Complementary Angles and Trigonometric Ratios. Load the question by clicking the button below.Angles H and Q are complementary. If $H=\frac{1}{12}Q$, calculate the following:

1. Find the values of the angles:
   (a) $H =$_______$°$
   (b) $Q =$_______$°$
2. Using a calculator, find the following correct to four significant figures:
   (a) sin $H =$_______
   (b) cos $Q =$_______

Answer 1. 7

Answer 2. 83

Answer 3. 0.1219

Answer 4. 0.9925

Solution. When we say H and Q are complementary, it implies that $H+Q=90°$. Also, we have $H=\frac{1}{12}Q$.

First, write H in terms of Q, that is, $H=\frac{Q}{12}Q=12H$

Since \({H}+{Q}=90^\circ, substitute $Q=12H$: $12H+H=90$ $13H=90$ $H=\frac{90}{13}$ $=7$

Therefore,

1. $\angle H=7°$
2. $\angle Q$

$$\begin{aligned}\angle Q&=90-7°\\&=83°\end{aligned}$$

Use your calculator to find both sin 7° and cos 7°, and write the answers correct to 4 significant figures. Therefore,

1. sin $7°=0.1219$
2. cos $83°=0.9925$

Checkpoint 2.4.23 Finding Sines and Cosines of Complementary Angles. Load the question by clicking the button below.Find the acute angles α and β that satisfy each of the following trigonometric equations:

1. cos $45°=\sin\alpha$

   $\alpha=$_______°
2. cos $\beta=\sin 5\beta$

   $\beta=$_______°
3. sin $2\alpha=\cos 30°$

   $\alpha=$_______°

Answer 1. 40

Answer 2. $\frac{90}{13}$

Answer 3. 0

Solution.

1. cos $45°=\sin\alpha$

   Implies that,

   $$\begin{aligned}50°+\alpha&=90°\\\alpha&=90°-50°\\&=40°\end{aligned}$$

   Therefore, The acute angle $\alpha=40°$
2. cos $\beta=\sin 5\beta$

   Implies that,

   $$\beta+12\beta=90°$$

$$13\beta = 90^\circ$$
$$\beta = \frac{90^\circ}{13}$$
$$= \frac{90}{13}^\circ$$

Therefore, The acute angle $\beta = \frac{90}{13}^\circ$

3. $\sin\, 2\alpha = \cos 30^\circ$

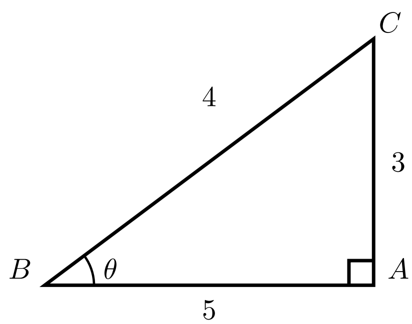
   Implies that,

$$2\alpha + 90^\circ = 90^\circ$$
$$2\alpha = 90^\circ - 90^\circ$$
$$\alpha = \frac{0^\circ}{2}$$
$$= 0^\circ$$

Therefore, The acute angle $\beta = 0^\circ$

Exercises

1. If θ is an acute angle such that $sin\,(\theta) = \frac{3}{5}$, find $cos(90^\circ - \theta)$.
2. In the right-angled triangle below, find $sin\,\theta$ and $cos\,\theta$, then verify that $sin\,\theta = cos\,\theta$.



3. Given that $cos(32^\circ) = 0.848$, find the value of $sin(58^\circ)$ without using a calculator.
4. A ladder leans against a wall, making a 65° angle with the ground. Find the height at which the ladder touches the wall if the ladder is $10\,m$ long.

2.4.3 Trigonometric ratios of Special Angles((30°, 45° and 60°))

2.4.3.1 Tangent, Cosine and Sine of 45°

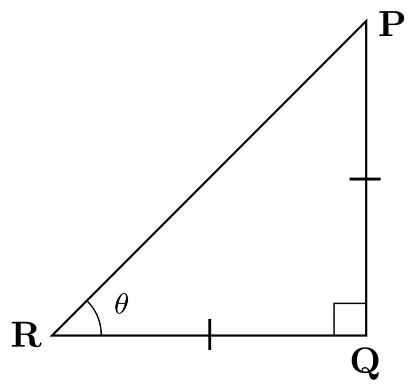
In this section you will use isosceles right-angle triangle to explore on how to find Tangent, Cosine and Sine of 45°

An **isosceles triangle** is a triangle whose two sides and base angles are equal.

Activity 2.4.7 Work in groups

What you require: A ruler, a piece of paper and a protractor.

1. Draw the isosceles right-angle triangle like the one below,(ensure that two sides are equal)



2. Measure the angle that substend the Hypotenues and the adjacent that is θ.

3. Measure the length PQ and QR

   What do you notice about PQ and QR

4. Find the following trigonometrc ratios:

   (a) $tan\,\theta$

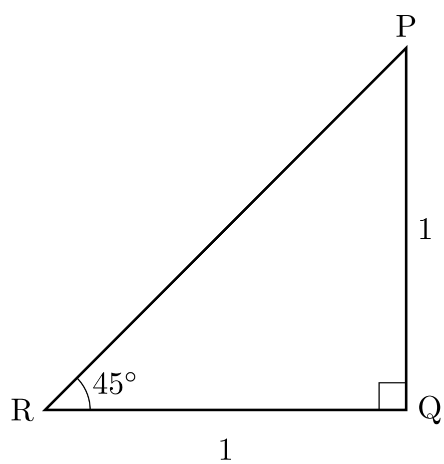
   (b) $cos\,\theta$

   (c) $sin\,\theta$

5. Compare the value of $cos\,\theta$ and $sin\,\theta$, How do they relate?

Key Takeaway

Look at the figure below;



To find the tangent, cosine and sine of 45°, find first the length PR

$$
\begin{aligned}
PR^2 &= 1^2 + 1^2 \\
&= 1 + 1 \\
&= 2 \\
PR &= \sqrt{2}
\end{aligned}
$$

Therefore,

$$\begin{aligned} Sin\,45^\circ &= \frac{1}{\sqrt{2}} \\ Cos\,45^\circ &= \frac{1}{\sqrt{2}} \\ Tan\,45^\circ &= 1 \end{aligned}$$

Checkpoint 2.4.24 Finding Trigonometric Ratios in a Right Angled Triangle (45°). Load the question by clicking the button below.

This question contains interactive elements.

Checkpoint 2.4.25 Determining the Length of a Divider Arm from Vertical Height. Load the question by clicking the button below.The angle formed by the arms of an upright pair of dividers and the horizontal is 45°. The vertical distance from the horizontal to the vertex is 13 cm. Find the length of one arm of the divider without using a calculator or trigonometric tables.

The Length of a divider arm from vertical height =_______ cm

Answer. $13\sqrt{2}$

Solution. ***Worked Solution:***

We are asked to find the length of one arm of the divider given:

- Angle with horizontal: 45°
- Vertical distance from horizontal to vertex: 13 cm

From trigonometry, for an upright divider forming an angle θ with the horizontal, the relationship between the vertical height and the arm length is:

$$\begin{aligned} \text{Arm length} &= \frac{\text{Vertical distance}}{\sin\theta} \\ &= \frac{13}{\sin(45^\circ)} \\ &= 13\sqrt{2} \text{ cm} \end{aligned}$$

Hence, the length of one arm of the divider is $13\sqrt{2}$ cm.

2.4.3.2 Tangent, Cosins and Sine of 30° and 60°

In this section, you will be using equilateral triangle to find the Tangent, Cosine and Sine of 30° and 60° .

Equilateral triangle is a triangle whose sides and angles are equal.

Work at home

Use the same procidure above to identify the Tangent, Cosine and Sine of 30° and 60° using equilateral triangle.

Discuss your work with other learners in your class.

Further activity

Activity 2.4.8 Work in groups

What you require: Ruler, pencil, protractor, graph paper.

1. Drawing a $45 - 45 - 90$:
   - (i) Draw a square.
   - (ii) Draw a diagonal to make two triangles.
   - (iii) Label angles (45°, 45°, 90°).

2. Draw $30-60-90$:

   (i) Draw an equilateral triangle.

   (ii) Draw a line from a corner to the middle of the opposite side.

   (iii) Label angles (30°, 60°, 90°).

3. Measure and Calculate:

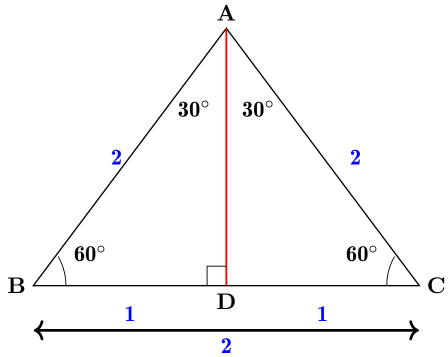
   - Measure sides of each triangle.
   - Calculate sin, cos, tan for $30°, 45°, 60°$ angles.
   - Record your results in the table below.

Angle	Triangle type	Opposite side length	Adjacent side length	Hypotenuse le
30°	$30-60-90$			
60°	$30-60-90$			
45°	$45-45-90$			

4. Discuss your result with other . What do you notice for the **Special Angles** $(30°, 45°, 60°)$?

Essential concepts.

The figure below shows an equilateral triangle ABC. AD Is the perpendicular bisector of BC.



Notice that;

length AD is given by,

$$\begin{aligned} AD^2 &= 2^2 - 1^2 \\ &= 4 - 1 \\ AD &= \sqrt{3} \end{aligned}$$

Therefore, $sin\,30° = \frac{1}{2}$, $cos\,30° = \frac{\sqrt{3}}{2}$, and $tan\,30° = \frac{1}{\sqrt{3}}$

Similarly,

$sin\,60° = \frac{\sqrt{3}}{2}$, $cos\,60° = \frac{1}{2}$, and $tan\,60° = \sqrt{3}$

Conclusion

- Sin and Cos are just swapped between 30° and 60°.
- Tan 30° is small while tan 60° is large.
- $sin\,\theta \times sin\,\theta$, can be written as $sin^2\,\theta$

Example 2.4.26 Simplify the following without using tables or calculator.

1. $sin\,30° \, cos\,45°$
2. $8\,cos\,45° \, sin\,45°$
3. $sin\,60° \, cos\,45° + sin\,30° \, tan\,45°$

Solution.

1.

$$\begin{aligned} sin\,30° \, cos\,45° &= \frac{1}{2} \times \frac{1}{\sqrt{2}} \\ &= \frac{1}{2\sqrt{2}} \end{aligned}$$

2.

$$\begin{aligned} 8\,cos\,45° \, sin\,45° &= \left(8 \times \frac{1}{\sqrt{2}}\right) \times \frac{1}{\sqrt{2}} \\ &= \frac{8}{\sqrt{2}} \times \frac{1}{\sqrt{2}} \\ &= \frac{8}{\sqrt{2} \times \sqrt{2}} \\ &= \frac{8}{2} \\ &= 4 \end{aligned}$$

3.

$$\begin{aligned} sin\,60° \, cos\,45° + sin\,30° \, tan\,45° &= \left(\frac{\sqrt{3}}{3} \times \frac{1}{\sqrt{2}}\right) + \left(\frac{1}{2} \times 1\right) \\ &= \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2} \\ &= \frac{\sqrt{3} + \sqrt{2}}{2\sqrt{2}} \end{aligned}$$

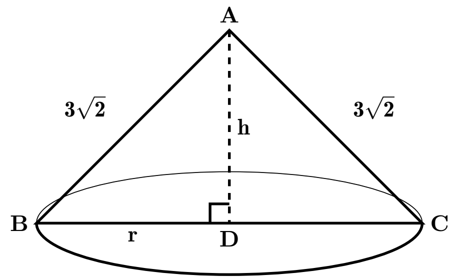
□

Example 2.4.27 The angle at the vertex of a cone is 90°. If the slant height is $3\sqrt{2}\,cm$. Find without using tables:

(a) The diameter of the cone

(b) The height of the cone.

Solution. Since the vertex angle is 90°, the cone can be thought of as half of a right circular cone, meaning that the base of the cone forms the hypotenuse of a right-angled triangle.



Let:

- r be the radius of the base,
- h be the height of the cone,
- $l = 3\sqrt{2}\,cm$ be the slant height (hypotenuse of the right-angled triangle).

Since the triangle formed is a right-angled isosceles triangle (because of the 90° vertex angle), we can say:
$r = h$
Since the right-angled triangle has radius r and height h, we use:
$r^2 = h^2 = l^2$
Since $r = h$, we substitute:

$$\begin{aligned} r^2 + r^2 &= \left(3\sqrt{2}\right) \\ 2r^2 &= 9 \times 2 \\ r^2 &= 18 \\ {}^2 &= 9 \\ r &= 3\,cm \end{aligned}$$

1. The diameter of the cone is:

   Diameter $= 2r = 2(3) = 6\,cm$

2. The height of the cone;

   Since $h = r$, we conclude:

   $h = 3\,cm$

□

Checkpoint 2.4.28 Finding the Trigonometry Ratios in a 30°, 60° and 90° Triangle. Load the question by clicking the button below.

This question contains interactive elements.

Exercises

1. The angle made by the arms of an upright pair of dividers and the horizontal is 45°. The vertical distance from the horizontal to the vertex is 15 cm. Find without using tables:

   (a) The horizontal distance between the tips of the arms.

   (b) The length of the arms.

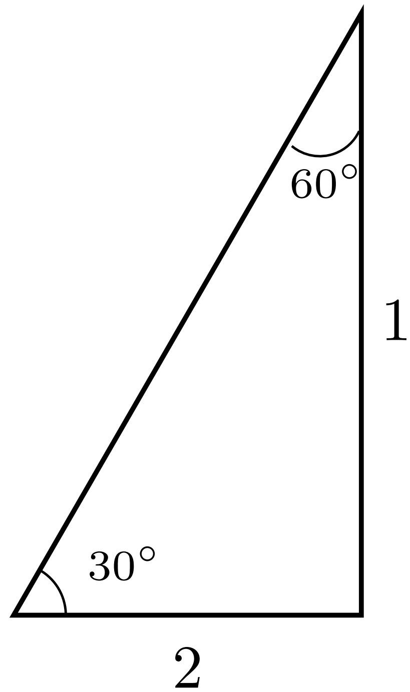
2. Without using a calculator, find

   $sin\,60° + cos\,30°$

3. Given that $\theta = 45°$, calculate:
   $tan^2\,\theta - sin^2\,\theta$

4. Using the triangle below, calculate the value of:

   a) $sin\,30°$
   b) $cos\,60°$
   c) $tan\,30°$



5. Given: $sin\,\theta = \frac{1}{2}$
   Find the angle θ, where $\theta° \leq \theta \leq 90°$

2.4.4 Determining Trigonometric Ratios Using a Calculator

Activity 2.4.9 Work in pairs
What you require:

- Scientific calculators or any other calculator having *sin*,*cos* and *tan* battons.
- Protractor and ruler (for optional verification with a drawn triangle).
- Worksheet with a table.

1. Turn on your scientific calculator.
2. Ensure your calculator is set to degree mode (not radians).
3. For each given angle (0°, 25°, 30°, etc.), do the following;
   - Press the sin button followed by the angle, then note the value.
   - Press the cos button followed by the angle, then note the value.

- Press the tan button followed by the angle, then note the value.

4. Record all values in the table below.

   - Use a calculator to find the sine, cosine, and tangent of the given angles, and fill in the table.

Angle (°)	*sin*	*cos*	*tan*
0°			
25°			
30°			
45°			
60°			
75°			
90°			

5. Observe and Answer

   - Observe the values of sin in your table. Do they increase or decrease?
   - Compare the cosine values for different angles.
   - Why does $tan\,90°$ display a syntax error?

6. Discuss your work with other learners.

Key Takeaway

When given an acute angle, a calculator can be used to determine these ratios accurately.

How to determine trigonometric ratios using a calculator.

1. Ensure the calculator is in degree mode.

- Press the "`MODE`" button and select "`DEG`" (if using a scientific calculator).

2. Enter the angle value.
   - For example, to find $sin\,30°$, type:
     - Press `sin` → Press `30` → Press `=`
     - The calculator should display 0.5 or some calculator will display $\frac{1}{2}$.

Example 2.4.29 Use calculator to find the following;(write your answer to **4 decimal places**).
$sin\,40°$ $cos\,40°$ $tan\,40°$

Solution. Ensure the calculator is in degree mode

1. $sin\,40°$

   Press `sin` → Press `40` → Press `=`

   $sin\,40° = 0.6428$

2. $cos\,40°$

   Press `cos` → Press `40` → Press `=`

   $cos\,40° = 0.7660$

3. $tan\,40°$

   Press `tan` → Press `40` → Press `=`

   $tan\,40° = 0.8391$

□

Example 2.4.30 Find $sin\,25°$ $cos\,25°$ $tan\,25°$ using calculator .

Solution.

- $sin\,25°$

  Press `sin` → Press `25` → Press `=`

  $sin\,25° = 0.4226$

- $cos\,25°$

  Press `cos` → Press `25` → Press `=`

  $cos\,25° = 0.9063$

- $tan\,25°$

  Press `tan` → Press `25` → Press `=`

  $tan\,25° = 0.4663$

□

Exercises

Use a calculator to determine the following trigonometric ratios:

1. Find $sin\,35°$ $cos\,35°$ $tan\,35°$.
2. Find $sin\,50°$ $cos\,50°$ $tan\,50°$.
3. Find $sin\,15°$ $cos\,15°$ $tan\,15°$.
4. Find $sin\,75°$ $cos\,75°$ $tan\,75°$.

Checkpoint 2.4.31 Use a calculator to compute the following sines and cosines.

1. $\sin(90°) =$ __________
2. $\sin(150°) =$ __________
3. $\cos(150°) =$ __________
4. $\cos(30°) =$ __________

Answer 1. 1

Answer 2. $\frac{1}{2}$

Answer 3. $-\frac{\sqrt{3}}{2}$

Answer 4. $\frac{\sqrt{3}}{2}$

Solution. In this case using the sin or cosine button on a calculator should give the correct evaluation.

Checkpoint 2.4.32 Using a calculator, calculate the sine, cosine, and tangent of 85.56°, giving your answers correct to 4 decimal places.
(i). $\sin 85.56° =$_____
(ii). $\cos 85.56° =$_____
(iii). $\tan 85.56° =$_____

Answer 1. 0.997

Answer 2. 0.0774

Answer 3. 12.8786

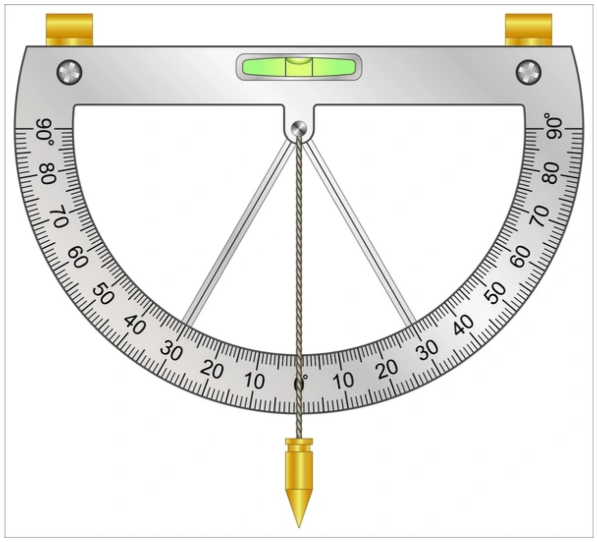
2.4.5 Application of Trigonometric Ratios

In this section you will learn how trigonometry are used in real life situation and many more.

2.4.5.1 Trigonometric Ratios to Angles of Elevation and Depression

Activity 2.4.10 Work in groups
What you require:

- Measuring tape or meter stick.
- Protractor or paper tube.
- Calculator (or table of trigonometric ratios).
- Paper and pencil.
- Use a clinometer, which is made from a string with a weight or rock attached, like the one shown below.



1. Find a tall object around your suraounding e.g tree, building, flagpole, etc.

2. Measure the distance from the base of the object to your position. Record this distance.

3. Use your clinometer to measure the angle of elevation from your eye level to the top of the object. Record the angle.

4. Draw a right triangle representing the situation. Label the distance you measured, the angle of elevation, and the unknown height of the object.

5. Decide which trigonometric ratio (sine, cosine, or tangent) you need to use to find the height.

6. Use the appropriate trigonometric ratio and the measured distance and angle to calculate the height of the object. Show your work

7. Write down your calculated height. Compare your result with your partner's or other groups' results. Discuss any differences.

NO	Object Observed (e.g tree)	Distance from Base (meters)	Angle of Elevation (de
1			
2			
3			

Extended Activity

Activity 2.4.11 Individual work

What you require: Your homemade clinometer, a ruler or measuring tape, a notebook and pen and a friend (optional, but fun!)

1. Climb up to a higher place like a step, a balcony, or a small hill

2. Hold the clinometer at eye level, and look through it toward an object on the ground (like a cone, stone, or your friend's shoes).

3. Watch the string and record the angle where it crosses the scale. That's your angle of depression!

4. Measure the height from your eyes to the ground (that's your vertical distance).

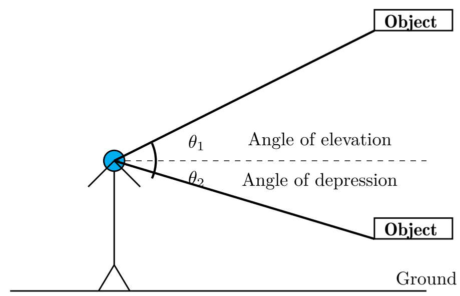
   Calculate how far the object is from the base of your standing point.

5. Share and discuss your work with your classmates.

Key Takeaway

A **clinometer** (or inclinometer) is a tool used to measure the angle of elevation (looking up) and the angle of depression (looking down).

The figure below shows a person standing on the ground, looking at an object at the top. This forms an angle of elevation. When looking down from a higher point, it forms an angle of depression.

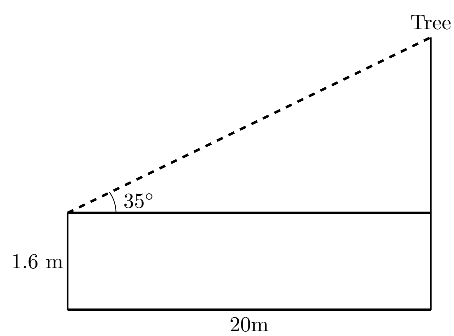


The dashed line is the horizontal line.

- **Angle of Elevation:** The angle measured upward from a horizontal line to an object above.
- **Angle of Depression:** The angle measured downward from a horizontal line to an object below.

Example 2.4.33 A person stands $20\,m$ away from a tree. The angle of elevation from their eyes is ($1.6m$ above the ground) to the top of the tree is $35°$. Find the height of the tree.

Solution. Look at the scetch below;



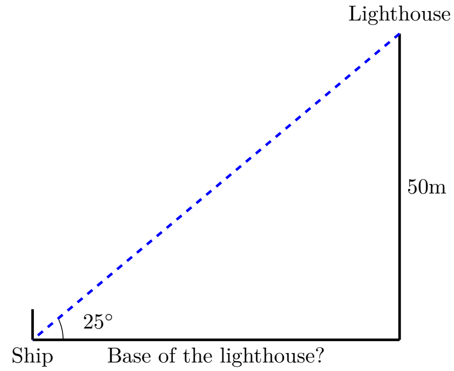
Use $tan\,\theta$

$$\begin{aligned} tan\,35° =& \frac{\text{Tree height} - 1.6\,m}{20\,m} \\ \text{Tree height} =& (20 \times tan\,35°) + 1.6 \\ =& (20 \times 0.7002) + 1.6 \\ =& 14 + 1.6 \\ =& 15.6\,m \end{aligned}$$

□

Example 2.4.34 A lighthouse is 50 m tall. A sailor spots the top of the lighthouse at an angle of elevation of °. How far is the ship from the base of the lighthouse?

Solution. Look at the scetch below;



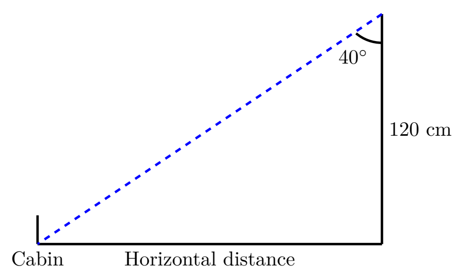
Use tan

$$\begin{aligned} tan\,25^\circ =& \frac{50\,m}{\text{Base}} \\ \text{Base} =& \frac{50}{tan\,25^\circ} \\ =& \frac{50}{0.4663} \\ =& 107.2\,m \end{aligned}$$

□

Example 2.4.35 A hiker stands on top of a hill that is 120 cm high and looks down at a cabin in a valley. The angle of depression to the cabin is 40°. Calculate the horizontal distance from the hiker to the cabin.

Solution. See the figure below;



Use tan:

$$\begin{aligned} tan\,40^\circ =& \frac{120\,cm}{\text{Cabin distance}} \\ \text{Cabin distance} =& \frac{120}{tan\,40^\circ} \\ =& \frac{120}{0.8391} \\ =& 143\,cm \end{aligned}$$

□

Exercises

1. A ladder is leaning against a wall, forming an angle of 60° with the ground. If the ladder is 10 meters long, how high does it reach on the wall? Draw a right-angled triangle to represent the situation

2. If the angle of elevation is 30° and the distance to the object is $50\,m$, then the height above eye level is:

3. A drone flies to a height of 80 meters above the ground. The angle of depression from the drone to a person standing on the ground is 30°. Find the horizontal distance between the person and the drone's projection on the ground. Sketch the problem.

4. A surveyor is standing 50 meters away from the base of a mountain. The angle of elevation to the peak of the mountain is 30°. Calculate the height of the mountain above the surveyor's eye level.

5. If the angle of depression is 40° and the horizontal distance to the object is $20\,m$, the vertical top is:

Checkpoint 2.4.36 This question contains interactive elements.

Checkpoint 2.4.37 This question contains interactive elements.

Technology Integration: Exploring Trigonometric Ratios

To learn more on trigonometric ratios, find these interactive and insightful resources:

1. Khan Academy – Interactive Learning

   Learn trigonometric ratios in right triangles through step-by-step tutorials and hands-on exercises.

   https://www.khanacademy.org/math/geometry/hs-geo-trig/hs-geo-trig-ratios-intro/a/finding-trig-ratios-in-right-triangles

2. Trigonometry Short Course Guide

   A well-structured PDF with in-depth explanations and examples to strengthen your grasp of trigonometry.

   https://www.govst.edu/uploadedFiles/Academics/Colleges_and_Programs/CAS/Trigonometry_Short_Course_Tutorial_Lauren_Johnson.pdf

3. YouTube Video – Visual Explanation

   Gain a clear and concise understanding of trigonometric ratios with this engaging video tutorial.

   https://youtu.be/uyKvSe6Ltgs

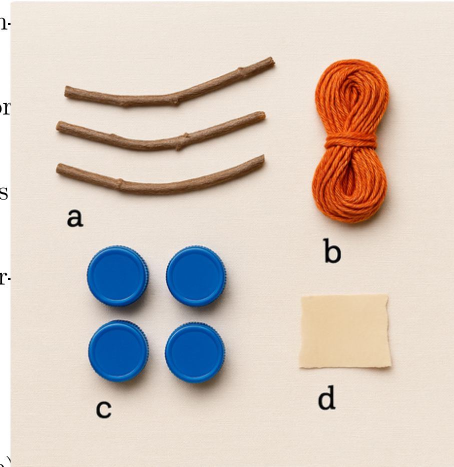
2.5 Area of Polygons

Why polygons?

The **area of a polygon** refers to the amount of space enclosed within its boundaries. It is measured in square units e.g., cm^2, m^2, in^2 etc.

Activity 2.5.1 Materials needed:

1. Sticks (e.g. matchsticks, toothpicks, broom straws, twigs)
2. Thread, string, rubber bands, or yarn
3. Recycled cardboard, cereal boxes or packaging paper
4. Bottle caps or buttons (as vertices)
5. Scissors
6. Glue or tape **(d)**
7. Ruler and protractor (if available)

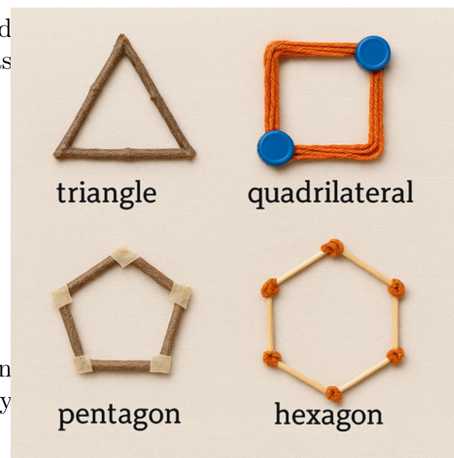


1. Have you seen shapes with straight sides around you?
2. What do you think makes a shape a polygon?

   In pairs or small groups, create physical models of different polygons using the available materials.

3. Using sticks **(a)** or straws for sides and caps**(c)** or buttons for vertices

1. Use sticks or straws for sides and caps or buttons for vertices as shown alongside.
2. (a) Construct at least:
   (b) A triangle
   (c) A quadrilateral
   (d) A pentagon
   (e) A hexagon
3. Present what type of polygon each model is and how many sides/vertices it has.
4. "What's common in all these shapes?"
5. "What's the minimum number of sides a polygon can have?"



Key Takeaway

A **polygon** is a closed two dimension formed by straight lines that connect end to end at a point called a **vertex**. Examples of polygons include triangles, quadrilaterals, pentagons, hexagons, heptagons,octagons, nonagons and decagons. **Their sides do not curve or overlap.**

- Note that most of the interior angles of a polygon are smaller than exterior angles.

  Types of Polygons

  We have different types of polygons but we are going to focus on only two in this chapter Regular polygons and irregular polygons.

Regular polygons – All sides and angles are equal (e.g., square, equilateral triangle).

Irregular polygons – Sides and angles are not equal (e.g., a scalene triangle, a random quadrilateral).

Convex polygons – All interior angles are less than 180°.

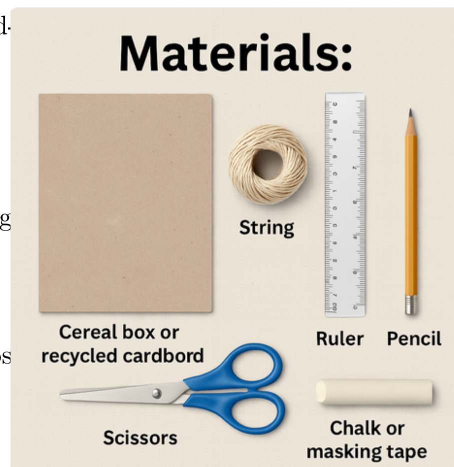
Concave polygons – At least one interior angle is greater than 180°.

Why is Area Important?
Would you wish to pursue a career some day? Have you ever seen an Architect?
So finding area is Essential in design and architecture as well as construction of houses for it helps to determine the spacing of rooms in a house in relation to the effect desired.

2.5.1 Area of Triangles.

Activity 2.5.2 Building a Triangle Garden.
Working im groups of groups of 3–5 students.
Materials needed:

- Cereal boxes or recycled cardboard
- String or yarn
- Pencil and Scissors
- Chalk (if outdoors) or masking tape (if indoors)
- Manila paper or newspaper
- A few small stones or bottle caps (for weights or markers)



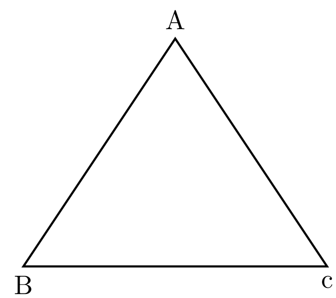
1. Preferably conuct this activity outside on sand or cement, or inside with tape/string.

    - Ensure you have a rectangle (or square) drawn with chalk/tape—e.g. 60 cm × 40 cm.
    - Can you use string to form a triangle that perfectly fits inside this rectangle?
    - Can you make the triangle cover exactly half of the space?
    - Use a string from corner to corner, or make triangles by connecting midpoints.
    - Take a manila paper or cardboard. Draw a rectangle (e.g. 10 cm by 6 cm).
    - Cut along a diagonal to get two right triangles. Rearrange the two triangles to form a rectangle again.
    - Now try : Drawing a triangle with the same base and height, but different shape (scalene, obtuse). Cut and rearrange it against the original rectangle.

2. On the board or chart, formulate the rule:
    - If the area of a rectangle is base × height, and the triangle is always half of it, What will be the formula of a triangle?
    - Write it down together in your groups.

Review of properties of a triangle.

- A triangle is a three-sided polygon with three angles which add up to 180° and three vertices.



- **Key Properties of a Triangle**

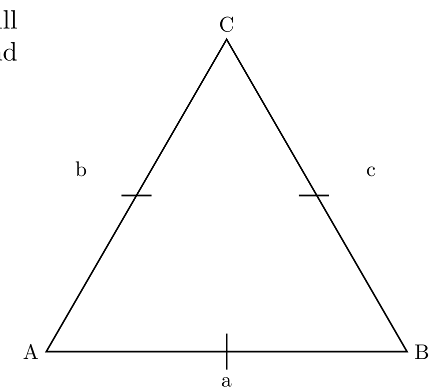
  A triangle has three Sides and Three Angles.

  A triangle has three edges and three vertices (corner points).

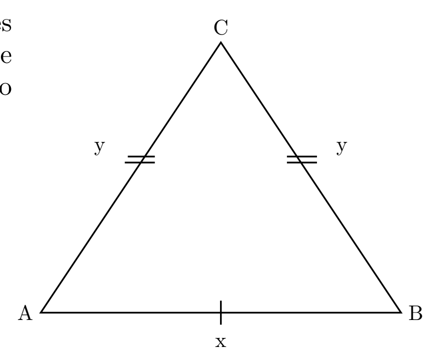
  Angle Sum Property - The sum of all three interior angles of a triangle adds up to 180°.

- **Types of Triangles (by Sides).**

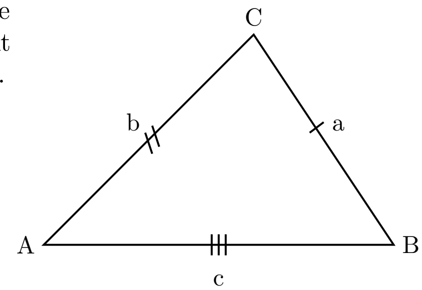
  Equilateral Triangle - All sides of the triangle are equal and have an angle of 60° each.



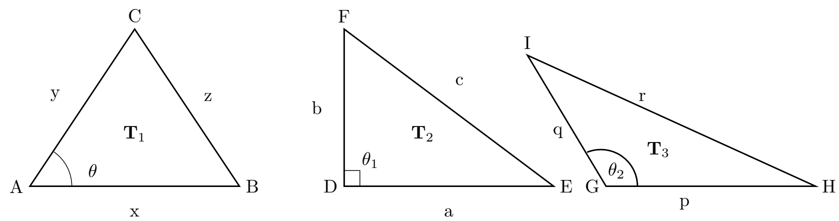
  Isosceles Triangle - Two sides of the triangle are equal, and the angles opposite these sides are also equal.



Scalene Triangle - All three sides of the triangle have different lengths, and all angles are different.



- Types of Triangles (by Angles).

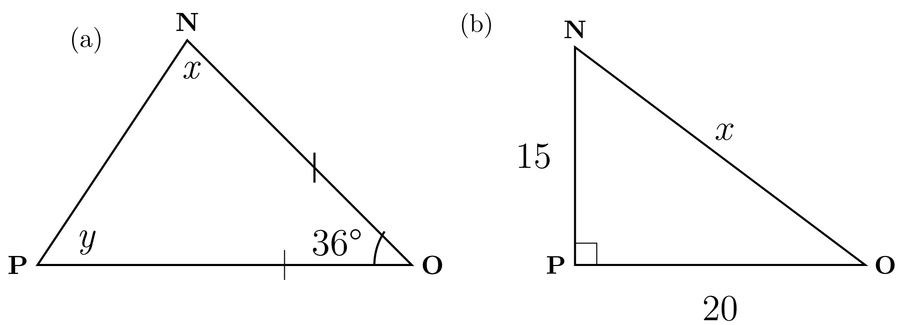


Acute-Angled Triangle(T_1) - All three angles are less than 90°.

Right-Angled Triangle(T_2) - One angle is exactly 90°.

Obtuse-Angled Triangle(T_3) - One angle is greater than 90° but less than 180 ∘ .

Example 2.5.1 Calculate the unknown variables in each of the following figures



Solution.

(a) An isosceles triangle has two angles that are equal. Sum of interior angles of triangle is 180°

$$=180° - 36°$$
$$=144°$$
$$=\frac{144°}{2}$$
$$=72°$$

Therefore $x = 72°$, $y = 72°$

(b) To find x we use pythagorean relationship.

$$a^2 + b^2 = c^2$$

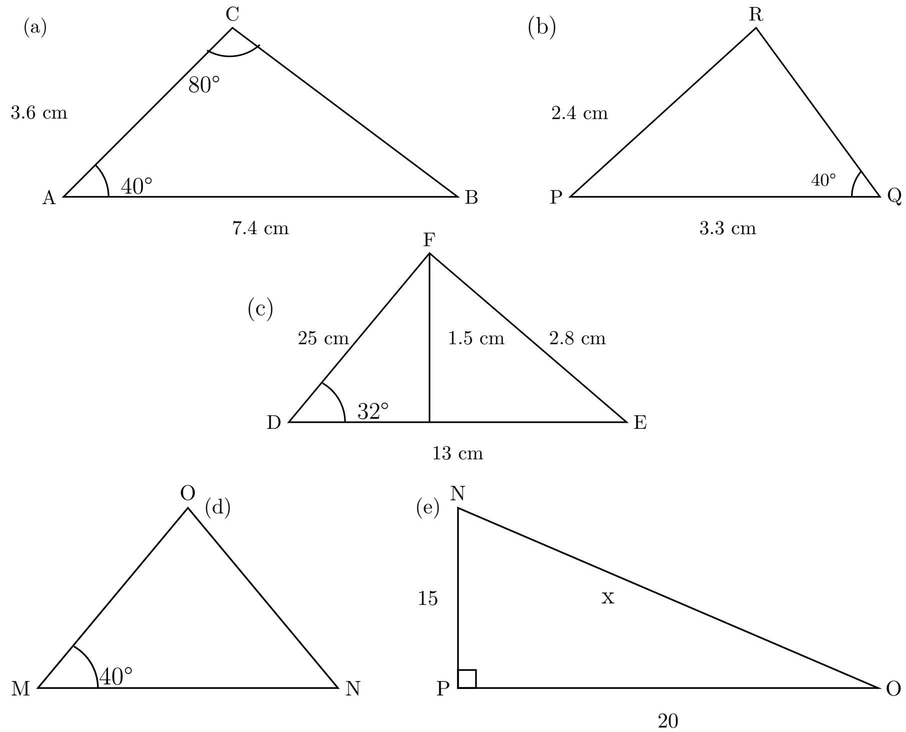
In our case our c is x

$$20^2 + 15^2 = c^2$$
$$400 + 225 = 625$$
$$= \sqrt{625}$$
$$= 25\,\text{cm}$$

□

Exercise

Determine the unknown angles and sides in the figures below



2.5.1.1 Area of a Triangle using Heron's Formula

Activity 2.5.3

Pair up students and give each pair a triangular surface to measure. This could be:

- A triangular classroom table
- A triangular signboard
- A triangular cutout from paper
- A drawn triangle on graph paper

  Materials needed.

- Ruler
- Graph paper
- Calculator
- String or measuring tape
- Pre-cut paper triangles (optional)

1. Recall the Standard Triangle Area Formula
   - How do we find the area of a triangle?
     Expected answer ; $a = \frac{1}{2} \times \text{base} \times \text{height}$
   - This method works only when we know the height.
   - **What if we don't know the height?** Today, they will discover how to find the area **only using side lengths**.
2. Construct a Triangle and Introduce Semi-Perimeter

- Draw a triangle with sides labeled as **a, b** and **c**. Using a ruler or measuring tape, measure the **three sides of the triangle** and record the lengths.
- Measure and calculate the semi-perimeter using:

$$S = \frac{a+b+c}{2}$$

  This helps divide the triangle into manageable parts for the area calculation.

3. Express the **Area** in Terms of the **Semi-Perimeter.**
   - Use the right-angle criteria.i.e:
     Drop a perpendicular from **one vertex to the opposite side** (splitting the triangle into two right-angled triangles).
     Let's call the base **b** and split it into two parts using the perpendicular.
   - Use the **Pythagorean Theorem** to express the height h in terms of **a, b, c**.
     Instead of solving fully, guide students to realize that the height can be found using algebraic manipulation.
   - Introduce the **squared form** and take the **square root**:

$$A = \sqrt{S(S-a)(S-b)(S-c)}$$

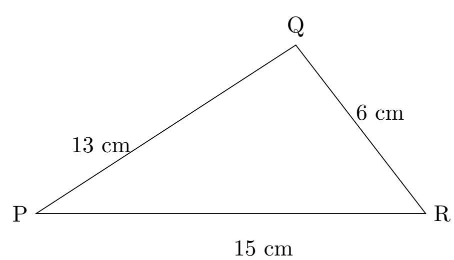
   - This is the final area using Heron's Formula.
   - In conclusion, Heron's Formula is finding the area of any triangle using only its sides.

Key Takeaway

Given three unequal sides of a triangle, we can calculate the area using a formula called Heron's Formula.

Follow the simple guidelines below and let's explore on how to get to the formula.

Consider the triangle below.



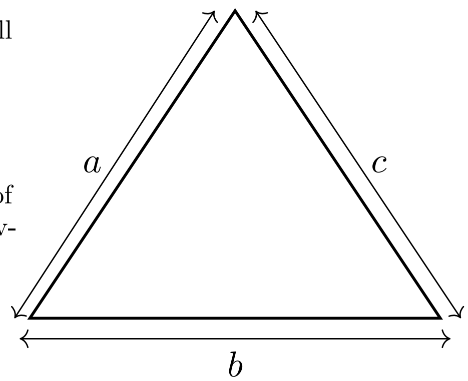
- Perimeter refers to adding all round.

$$\mathbf{P} = a + b + c$$

- Let's find the Semi perimeter of the triangle by the perimeter diving by half.

$$\mathbf{S} = \frac{a+b+c}{2}$$

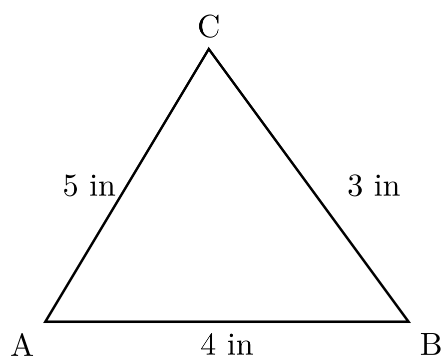
- Then find the area using Heron's formula:



$$\mathbf{A} = \sqrt{S(S-a)(S-b)(S-c)}$$

- ◦ Calculate **S** (half of the triangle's perimeter):
- ◦ Then calculate the Area:

Example 2.5.2 If the length of the sides of a triangle ABC are 4 *inches*, 3 *inches* and 5 *inches*. Calculate its area using the heron's formula.



Solution. To find: Area of the triangle ABC.
Given that $AB = 4$in and $BC = 3$in, $AC = 5$ in.
Using Heron's Formula,

$$A = \sqrt{(s(s-a)(s-b)(s-c))}$$

$$s = \frac{(a+b+c)}{2}$$

$$s = \frac{(4+3+5)}{2}$$

$$= 6\,\text{units}$$

Substitute in the values,

$$A = \sqrt{(6(6-4)(6-3)(6-5))}$$

$$A = \sqrt{(6(2) \times (3) \times (1))}$$

$$A = \sqrt{36}$$

$$= 6\,\text{in}^2$$

The area of the triangle is $6\,\text{in}^2$. □

Exercise

1. In the diagram below $\angle ABC$ is right-angled at B. Complete the table below.

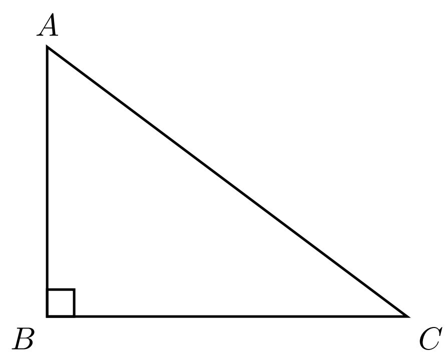
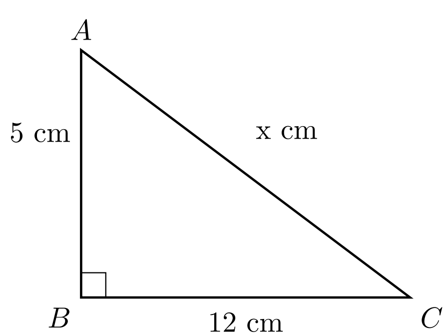


Table 2.5.3 Trigonometry question test

Trigonometic ratios	**Solution**
sin C	=
=	$\frac{BC}{AC}$
tan θ	=
Value of AC using pythagorean relationship	=
Area given a, b and sin θ	=

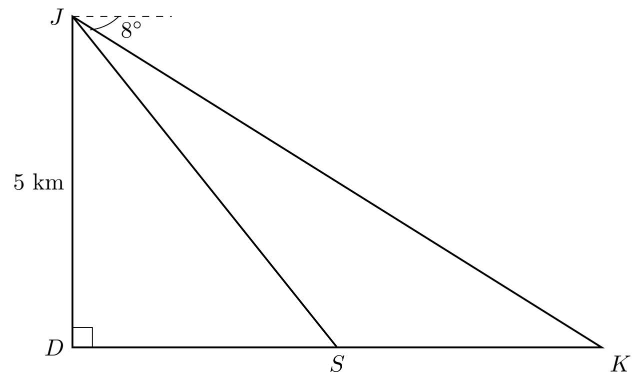
2.The area of a triangle with sides 5 cm, 12 cm, and x cm is 30 cm². Find the possible values of x.



3. A right-angled triangle has sides 9 cm, 12 cm and 15 cm.
(a) Verify that the triangle satisfies the Pythagorean relationship.
(b) Find its area using both the standard formula and Heron's formula.

4. A triangular plot of land has sides measuring 50 m, 65 m and 75 m. Find the area of the land using Heron's formula.

5. An aeroplane at J is flying directly over a point D on the ground at a height of 5 kilometres. It is heading to land at point K. The angle of depression from J to K is 8°. S is a point along the route from D to K.



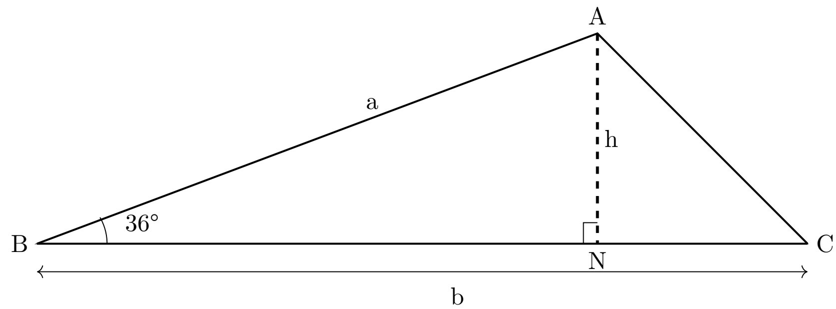
I) What's the size of $\angle JKD$?
II) Calculate the distance DK, correct to the nearest metre
III) If the distance SK is 8 kilometres, calculate the distance DS.
IV) Calculate the area of $\triangle JKD$.

2.5.1.2 Area of a triangle given two sides and an included angle

Activity 2.5.4

1. What you require:
   - A tall object (flagpole, tree or building)
   - Measuring tape or ruler.
   - A sunny day
2. **Identify the Tall Object** .
   - Choose a tree, flagpole, or lamp post as the vertical height (like A in the diagram below).
   - The ground acts as the base (BC).
3. **Find the Shadow Length**
   - Measure the length of the shadow cast by the object on the ground (BC).
   - This represents the long horizontal base in the diagram.
4. **Measure the Angle of Elevation**
   - Stand some distance away and use a protractor or a phone app to measure the angle of elevation from your eyes to the top of the object.
   - This represents the 30° angle at B in the diagram.
   - If no protractor is available, use similar triangles by measuring the shadow of a known object (like a stick) and comparing proportions.
5. Discuss your findings with your group members.

6.



- **Apply Trigonometry Using Sine**
  Using the sine function to calculate the height of the object AN.
  Drop a perpendicular line from Point A to meet line BC at N .
- Formula:

$$Height(AN) = \text{Hypotenuse(AB)} \times \ \sin 30^\circ$$

- AN is therefore the height for triangle ABC.
- In a triangle $\sin\theta = \frac{\text{Opposite}}{\text{Hypotenuse}}$.
  From the figure shown above, Opposite = AN, the height of triangle ABN whereas the hypotenuse i AB is the longest side of a triangle.
  Where $AN = \mathbf{h}$ and $BA = \mathbf{a}$

$$\text{In triangle ABN, sin}30^\circ = \frac{AN}{a}$$
$$\sin 30^\circ = \frac{\mathbf{h}}{\mathbf{a}}$$
$$\text{height of } \angle\text{ABC}\, \mathbf{h} = \mathbf{a} \sin 30^\circ$$
$$\text{Since } \sin 30^\circ = 1/2\mathbf{a}$$

- Area of Triangle ABC will be:

$$\triangle\, ABC = \frac{1}{2} \times \text{Base } \ (\mathbf{b}) \times \ \text{Height}\, (\mathbf{h}) = \mathbf{a}\sin\theta$$
$$= \frac{1}{2} \ \mathbf{ab} \ \ \mathbf{sin} \ \ \theta$$

Generally, if the lengths of the two sides and an included angle of a triangle are given, then the area of the triangle is A $= \frac{1}{2}\mathbf{absin}\theta$. Where **b** is the base and **h** height.

Exploration 2.5.5

1. Deriving the Formula: $\frac{1}{2}ab\sin$ C

   - Step 1: Let's recall the Basic Formula for finding area of a triangle.
   - Area $= \frac{1}{2} \times \text{Base} \times \text{Height}$

2. **Step 2: Consider a Triangle with an Angle**.

   - Let's take a triangle ABC with sides a, b and included angle C.
   - Side a and Side b form the triangle.
   - The height h is perpendicular from the top vertex to the base.

3. **Step 3: Express Height in Terms of Sin**
   - Using trigonometry, we know that in a right-angled triangle:
   - $\sin\theta = \dfrac{\text{Hypotenuse}}{\text{Opposite}}$
   - In our case, the height h is the **opposite side** of angle C, and side b is the **hypotenuse**.
   - So, we can write it as:
     $h = b\ \sin C$

4. **Step 4: Substitute into the Area Formula**
   - Now, take the basic Formula for finding area of a triangle;
   - Substituting **Base** and **Height** = b sin C .

$$\text{Area} = \frac{1}{2} \times a \times (\text{b sin C})$$
$$\text{Area} = \frac{1}{2} ab\ \text{ sin C}$$

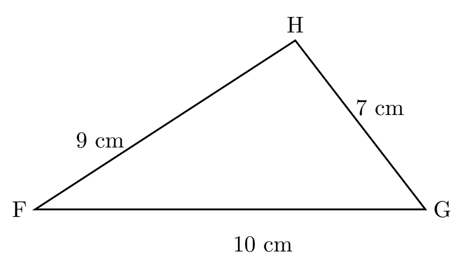
5. **NOTE:**
   - $$\text{Area} = \frac{1}{2} ab\ \text{ sin C}$$
   - This formula is useful when we know two sides of a triangle and the angle between them instead of the height..

Example 2.5.4

1.A triangle HFG has sides 10 cm, 7 cm and 9 cm. Find:
(a) Its area.
(b) The sizes of its angles.

2. The area of triangle ABC is 28.1cm^2 . It's side AB = 7.2cm and ∠ ABC = 48.6°. Find:
(a) The length of the perpendicular from A to BC.
(b) The length of BC.

Solution 1. (a) Find the area using Heron's formula.



$$A = \sqrt{(s(s-a)(s-b)(s-c))}$$

$$s = \frac{(a+b+c)}{2}$$

$$s = \frac{(10+7+9)}{2}\, cm$$

$$= 13\,\text{cm}$$

Substitute in the values.

$$A = \sqrt{(13(13-10)(13-7)(13-9))}$$

$$A = \sqrt{(13(6)\times(3)\times(4))}$$

$$A = \sqrt{936}$$

$$= 30.6\,\text{cm}^2$$

The area of the triangle is 30.6cm^2.

(b) Find the angles using the sine area formula.

$$A = \frac{1}{2}\,\text{ab sin C}$$

$$A = 30.6\ \ \text{cm}^2$$

To find $\angle HFG$ Using sides $a = 10$ and $b = 7$ and area= $30.6\,\text{cm}^2$

$$\begin{aligned} 30.6\,\text{cm}^2 =& \frac{1}{\not{2}} \times \not{10}\,5\,\text{cm} \times 7\,\text{cm} \times \sin A \\ \text{Sin A} =& \frac{30.6\not{\text{cm}^2}\ \text{cm}}{35\not{\text{cm}}} \\ \text{sin A} =& 0.8743 \\ A =& sin^{-1}\,0.8743 \\ =& 60.96^\circ \end{aligned}$$

To find $\angle FGH$ Using sides $a = 10$ and $b = 9$and area= $30.6\,\text{cm}^2$

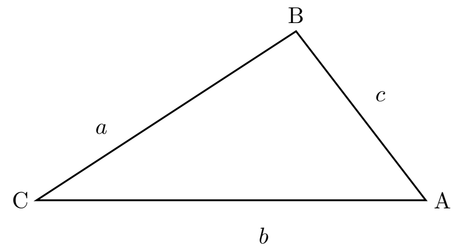
$$\begin{aligned} 30.6\,\text{cm}^2 =& \frac{1}{\not{2}} \times \not{10}\,5\,\text{cm} \times 9\,\text{cm} \times \sin C \\ \text{Sin C} =& \frac{30.6\not{\text{cm}^2}\ \text{cm}}{45\not{\text{cm}}} \\ \text{sin C} =& 0.68 \\ C =& \sin^{-1} 0.68 \\ =& 42.84^\circ \end{aligned}$$

To find the $\angle GHF$ we use the angle sum property.

$$\begin{aligned} \angle GHF =& 180^\circ - (\angle FGH + \angle HFG) \\ =& (180 - 103.80)^\circ \\ =& 76.2^\circ \end{aligned}$$

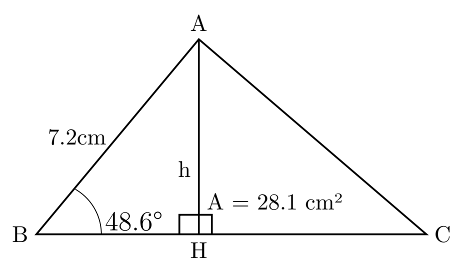
Similarly we can use the Sin A rule which states that :

$$\frac{\text{sin A}}{a} = \frac{\text{sin B}}{b} = \frac{\text{sin C}}{c}$$



$$\begin{aligned}
\frac{\text{sin B}}{b} &= \frac{\text{sin C}}{c} \\
\frac{\text{sin B}}{10\,\text{cm}} &= \frac{\text{sin } 0.8743}{9} \\
\sin B &= 0.9714 \\
&= \sin^{-1} 0.9714 \\
B &= 76.2^\circ
\end{aligned}$$

Solution 2. (a) Find the length of the perpendicular from A to BC.



The formula for area using base and height.

$$\begin{aligned}
A &= \frac{1}{2} \times base \times height \\
28.1\text{cm}^2 &= \frac{1}{2} \times b \times h \\
h &= \frac{2 \times 28.1}{BC}
\end{aligned}$$

(b) Finding BC using sine rule

$$\begin{aligned}
A &= \frac{1}{2} \times AB \times BC \times \sin\theta \\
28.1\,\text{cm}^2 &= \frac{1}{2} \times 7.2\,\text{cm} \times BC \times \text{sin } 48.6^\circ \\
\sin 48.6^\circ &= 0.7501 \\
28.1\,\text{cm}^2 &= \frac{1}{2} \times 7.2\,\text{cm} \times BC \times \sin 0.7501 \\
BC &= \frac{28.1 \times 2}{7.2 \times 0.7501} \\
&= 10.4\,\text{cm}
\end{aligned}$$

Now substituting BC into the height equation :

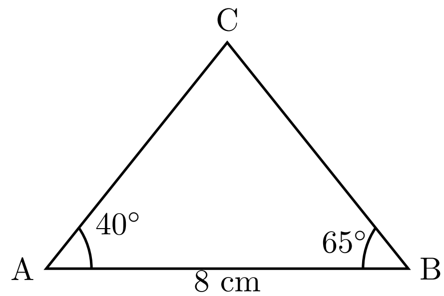
$$\begin{aligned} h =& \frac{2 \times 28.1}{10.4} \\ =& 5.4\,\text{cm} \end{aligned}$$

□

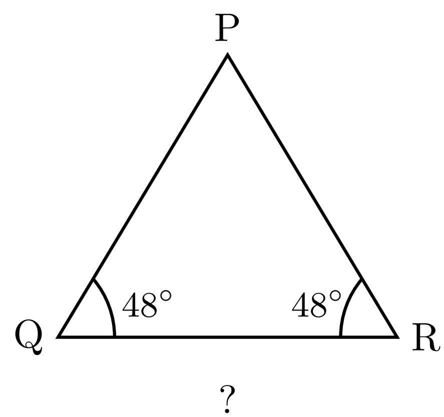
Exercise2

1. In a triangle $\triangle$**QRS**, **QR** = 10 cm, **RS** = 24 cm and **QS** = 26 cm. Find the length of the perpendicular from vertex **Q** to side **RS** .Therefore find it's area.

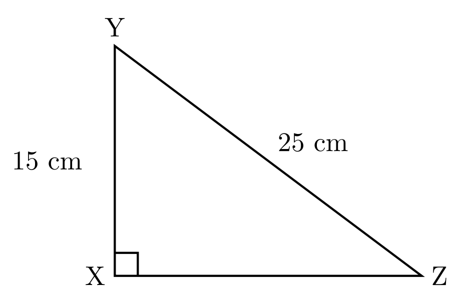
2. In $\triangle$**ABC**, $\angle$**BAC** = 40°, $\angle$**ABC** = 65°, and side **BC** = 8cm. Find it's area.



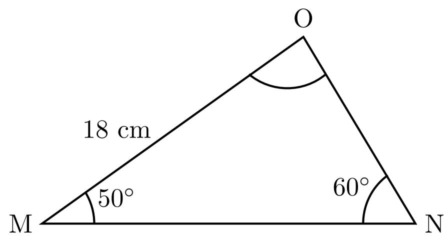
3.$\triangle$**PQR** is **isosceles** with **PQ** = **PR** = 10 cm. The base angle is 48°.Find its area.



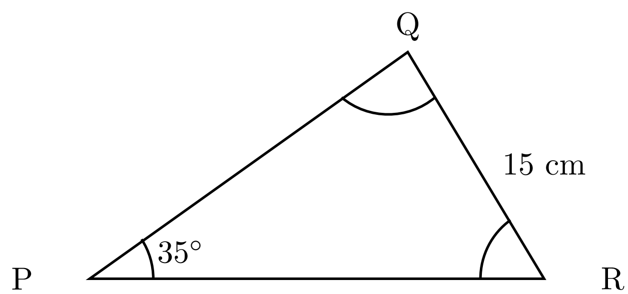
4. Triangle **XYZ** has side **XY** = 15 cm, **YZ** = 20 cm, and **XZ** = 25 cm. Show that this is a right-angled triangle and find it's area.



5. Triangle **MNO** has $\angle$**NMO** = 50°, $\angle$**MNO** = 60°, and **MO** = 18 cm. Find the length of **MN** using the sine rule thus find the triangle's area.



6.△**PQR** has ∠**RPQ** = 35°, ∠**PQR** = 75°, and **QR** = 15 cm. Find the area of triangle PQR.



2.5.2 Area of Quadrilaterals.

Activity 2.5.6

- **Materials needed:**
- Grid paper or geoboards
- Markers or colored pencils for highlighting shapes
- Printable worksheets (with grid paper and templates for quadrilaterals)
- Rulers, Pencils and erasers
- Protractors (for measuring angles)
- Scissors (if cutting shapes from paper)
- 1. Prepare the worksheets with grid paper or geoboard templates and space for calculations.
  2. Prepare cut-out shapes of different quadrilaterals (square, rectangle, parallelogram, rhombus, trapezoid, etc.), if using physical materials.
  3. Set up a projector or board to display instructions, the properties of quadrilaterals, and the step-by-step breakdown.
- Take that cut-out shapes or access to digital templates of various quadrilaterals. Sort the shapes based on their properties:

  Number of parallel sides

  Number of equal sides

  Angles (right angles or acute/obtuse angles)

  Symmetry
- For each quadrilateral, discuss with your patner and record:

  1. The type of quadrilateral.

2. Whether it has parallel sides or right angles.
3. The number of sides of equal length.

Learners should be able to classify quadrilaterals by their properties.

Key Takeaway

Properties of Quadrilaterals.

Below are the key ***types of quadrilaterals*** and their ***properties***:

Quadrilateral	Properties
Parallelogram	• Opposite sides are ***parallel*** and ***equal***. • Opposite angles are ***equal***. • Diagonals ***bisect each other***.
Rectangle	• Opposite sides are ***equal and parallel***. • All angles are $90°$. • Diagonals are ***equal and bisect each other***.
Square	• All sides are ***equal***. • All angles are $90°$. • Diagonals are ***equal, bisect each other at*** $90°$***, and are perpendicular***.
Rhombus	• All sides are ***equal***. • Opposite angles are ***equal***. • Diagonals ***bisect each other at*** $90°$.
Trapezium (Trapezoid)	• One pair of opposite sides is ***parallel***. • Non-parallel sides are called ***legs***. • If legs are equal, it is called an ***isosceles trapezium***.
Kite	• Two pairs of ***adjacent*** sides are ***equal***. • One pair of opposite angles is ***equal***. • Diagonals ***intersect at*** $90°$, and one diagonal bisects the other.

Can you remember the formula for finding the area of any Quadrilateral in the table above?

If yes which one and what is the formula?

Kindly write it down exchange your answers with your deskmate and see if their answers are correct.

Example 2.5.5 Find the area of a rectangular classroom measuring 6m by 4m .

Solution.

$$\begin{aligned} \text{Area of a rectangle} &= Length \times Width \\ &= 6\text{m} \times 4\text{m} \\ &= 24\text{m}^2 \end{aligned}$$

□

2.5.2.1 Area of a Parallelogram

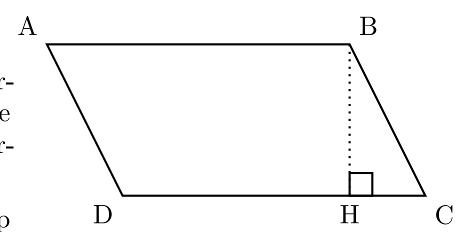
Activity 2.5.7 Exploring the Area of a parallelogram.

Materials needed.

- Graph paper
- Scissors
- Ruler
- Colored pencils
- A printed or drawn parallelogram template.

Steps for the activity.

1. Draw a parallelogram.



   (a) On graph paper, draw a parallelogram with a given base and height. Label the vertices A, B, C and D.

   (b) Use a pair of campus to drop a perpendicular. Lable the intersection point on line CD as H representing the height of the parallelogram.

   (c) Use a ruler to measure and compare opposite sides (AB and CD, BC and AD.)

Key Takeaway

A parallelogram is a quadrilateral whose opposite sides are equal and parallel. Which other quadrilateral has similar characteristics like these?

The area of a parallelogram is given by: $Base \times Height$:

$$\text{A} = b \times h$$

where b is the base and h the perpendicular distance between the given pair of parallel sides.

The area of a parallelogram given the base is 28cm and the height 8cm is:

$$\begin{aligned} \text{A} &= b \times h \\ &= 28\text{cm} \times 8\text{cm} \end{aligned}$$

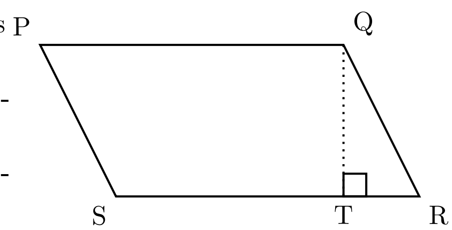
$$=224\,\text{cm}^2$$

Example 2.5.6

- A parallelogram PQRS is of sides 28cm and 7cm. If ∠ QRS is 75°.

  a. Find the height the parallelogram using sine rule.

  b. Find the area of the parallelogram.



Solution.

From Q drop a perpendicular to meet RS at T considering $\angle QRS$.
Height (QT) = Hypotenuse(BC) × sinθ

$$\begin{aligned} QT =& 7\sin 75^\circ \\ =& 6.76\,\text{cm} \end{aligned}$$

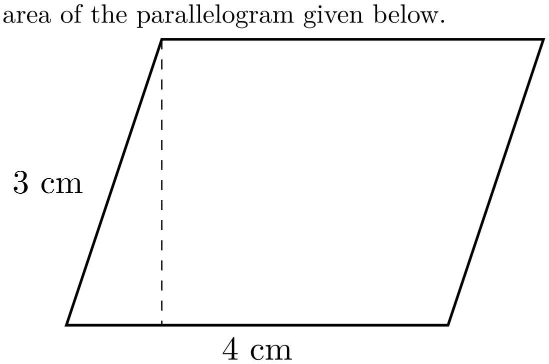
b.

$$\begin{aligned} \text{Area of PQRS} =& RS \times PT \\ =& 28 \times 7\sin 75^\circ \\ =& 189.32\,\text{cm}^2 \end{aligned}$$

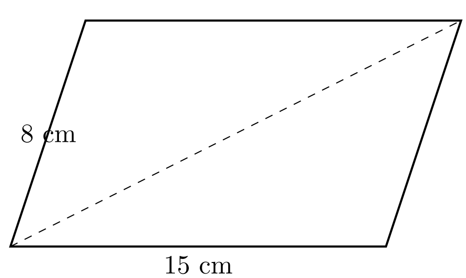
The area of the parallelogram is $189.32\,\text{cm}^2$. □

Exercise

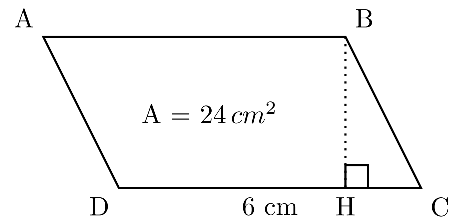
1. Find the area of the parallelogram given below.



2. The parallelogram shown below has diagonals. Find the length of the diagonal.



3. Find the missing height of the parallelogram ABCD given the area is 24 cm² and base is 6 cm.



4. A construction company is building a parallelogram-shaped floor with a base of 10 meters and height of 6 meters. Find the area of the floor.

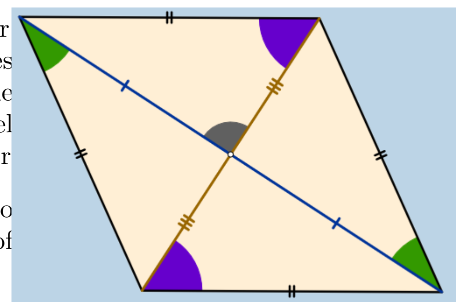
5. A billboard has a parallelogram shape with a base of 12 feet and a height of 5 feet. Find the area of the billboard.

6. A solar panel is shaped like a parallelogram with a base of 8 meters and an inclined height of 4 meters. Find it's surface area.

2.5.2.2 Area of a Rhombus

Activity 2.5.8 Materials Needed:

Ruler, Protractor, Pencil, Graph paper, Scissors (optional) and String or thread (optional)



- Draw a Rhombus Using a ruler draw a quadrilateral with all sides equal in length. Make sure the opposite angles are equal. Label the vertices as A, B, C, D in order
- Measure the Sides using a ruler to confirm that all four sides are of equal length.
- Measure the angles using a protractor to measure each of the interior angles. Observe: Opposite angles should be equal.
- Draw the Diagonals:

  Draw diagonal AC and diagonal BD.

  Measure their lengths and angles at the intersection point.

  Observe: Diagonals bisect each other at 90°, and they are not equal.
- Cut and Fold (Optional):

  Cut out the rhombus and fold it along both diagonals.

  Observe the symmetry and how the diagonals act as lines of symmetry.

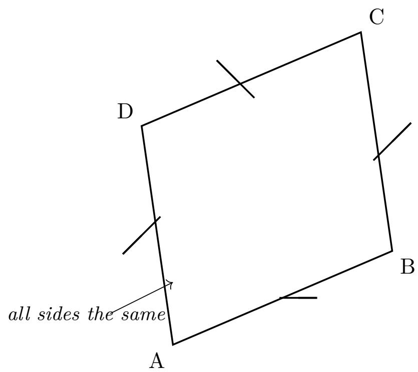
Discussion Questions

1. What do you notice about the sides and angles of the rhombus?

2. How do the diagonals interact with each other?
3. What makes a rhombus different from a square or a general parallelogram?

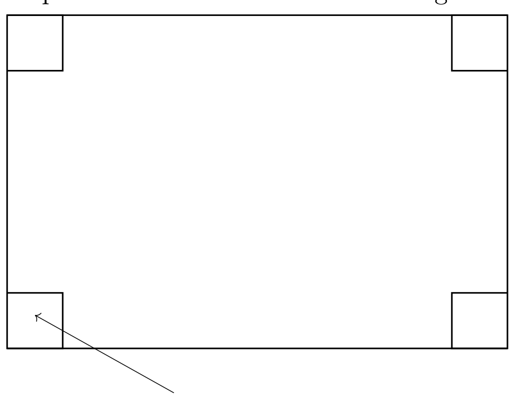
Key Takeaway

A **rhombus** is an equilateral quadrilateral. All it's sides are of **equal length**, But all it's angles are not. All its sides are equal and the diagnols bisect at 90°. Which other quadrilateral has all its sides equal?



Equilateral
A rhombus is an equilateral quadrilateral.
All its sides are equal, but
all its angles are not.

Compare the rhombus with the rectangle below.



all angles the same

Equiangular

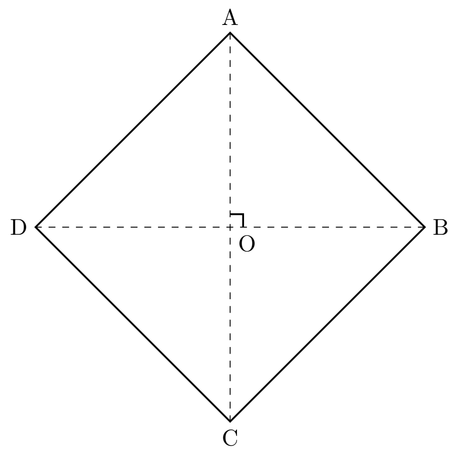
- A rectangle is an equiangular quadrilateral.
-Its angles are all equal, but not all its sides are equal.

Properties of a Rhombus

- All sides are equal in length.
- Opposite angles are equal.
- Diagonals bisect each other at right angles.
- Diagonals bisect the interior angles.

- Some examples of a rhombus are Tiles or patterns in flooring, or The shape of playing cards (diamonds suit).

Example 2.5.7 A rhombus has a diagonal of 10 cm and another diagonal of 24 cm. Find:
i) The area of the rhombus.
ii).The length of one side of the rhombus.



Solution. i.
One diagonal = 10 cm
Another diagonal = 24 cm

$$\begin{aligned}\text{Area} =&\frac{1}{2} \times d_1 \times d_2\\ =&\frac{1}{2} \times 10\,\text{cm} \times 24\,\text{cm}\\ =&120\,\text{cm}^2\end{aligned}$$

ii. Length of One Side of the Rhombus
In a rhombus, the diagonals bisect each other at 90°. That means each side of the rhombus forms a right-angled triangle with half of each diagonal.
Let's find the length of one side using the Pythagorean Theorem.
Each side of the rhombus is the hypotenuse of a right triangle with legs:

$$\begin{aligned}\text{Half of AC } =&\frac{10}{2} = 5\,\text{cm}\\ \text{Half of BD } =&\frac{24}{2} = 12\,\text{cm}\\ \text{Side} =&\sqrt{(5)^2 + (12)^2}\\ =&\sqrt{25\,\text{cm} + 144\,\text{cm}}\\ =&\sqrt{169\,\text{cm}}\\ =&13\,\text{cm}\end{aligned}$$

□

Exercise

1. A rhombus has diagonals measuring 16 cm and 30 cm.

   a) Find the area of the rhombus.

b) Find the side length of the rhombus.

2. A diamond-shaped road sign is a rhombus with a diagonal of 40 cm and another diagonal of 60 cm. Find:
   (a) The area of the sign.
   (b) The length of one side of the sign.
3. The perimeter of a rhombus is 48 cm. Find the length of one side.
4. A car logo is shaped like a rhombus. The diagonals measure 14 cm and 10 cm. Find its area.
5. A rhombus has one of its angles measuring 60°. Find the other three angles.
6. A tiling design on a floor is made of rhombus-shaped tiles. Each tile has diagonals of 18 cm and 24 cm.
   (a) Find the area of one tile.
   (b) If 20 such tiles cover a portion of the floor, what is the total area covered?

Checkpoint 2.5.8 An error occurred while processing this question.

Checkpoint 2.5.9 The perimeter of a rhombus is 1620 m. One of its diagonals is 486 m. Answer the following:

1. Find the length of the other diagonal.
   The other diagonal = ________ m
2. Find the area of the rhombus.
   Area = _______ m^2

Answer 1. 648

Answer 2. 157464

Solution. To answer the above question follow the following steps

Step 1: Find the length of one side

A rhombus has 4 equal sides. Therefore,

$$\begin{aligned} \text{Perimeter} &= 4 \times \text{side} \\ 1620 &= 4 \times \text{side} \\ \text{side} &= \frac{1620}{4} \\ &= 405\,\text{m} \end{aligned}$$

Step 2: Use the diagonal property

The diagonals of a rhombus cut each other into two equal parts and form right angles. • One diagonal = 486 m, so half of it = $\frac{486}{2} = 243$. • Let the other diagonal = d_2. Then half of it = $\frac{d_2}{2}$.

Using Pythagoras' Theorem:

$$\begin{aligned} 405^2 &= \left(\frac{486}{2}\right)^2 + \left(\frac{d_2}{2}\right)^2 \\ \left(\frac{d_2}{2}\right)^2 &= 405^2 - (243)^2 \\ d_2 &= 2 \times \sqrt{405^2 - (243)^2} \end{aligned}$$

So, the other diagonal is:

$$d_2 = 648\,\text{m}$$

Step 3: Find the area of the rhombus
The formula for the area is: Area = $\frac{d_1 \times d_2}{2}$ Therefore;

$$\text{Area} = \frac{486 \times 648}{2}$$
$$= 157464\,\text{m}^2$$

Therefore: The length of the other diagonal is: 648 m The area of the rhombus is: $157464\,\text{m}^2$

2.5.2.3 Area of a Trapezium

Activity 2.5.9
materials needed:
Materials: Ruler, Protractor, Pencil, Graph paper, Scissors (optional) and Calculator (optional)

- Draw a Trapezium:

  On graph paper, draw a quadrilateral with one pair of opposite sides parallel (e.g., AB CD).

  Make sure the other pair (AD and BC) are not parallel.

  Label the trapezium as ABCD.

- Identify the Parts:

  Label the bases (parallel sides).

  Label the legs (non-parallel sides).

  Draw the height (perpendicular distance between the two bases).

- Now, let's break it down algebraically:

  Imagine the Area of a Rectangle: If we could transform the trapezium into a rectangle, the area of that rectangle would be the **average of the two bases times the height** . This works because the average length of the two parallel sides is a **"representative" length for the trapezium,** and multiplying it by the **height** gives us the **correct area**.

- Mathematical Expression:

  The average length of the two parallel sides is given by

$$\text{Average of bases} = \frac{Base_1 + Base_2}{2}$$

  Therefore, the area of the trapezium is:

$$Area = \text{Average of bases} \times \text{height}$$
$$= \frac{Base_1 + Base_2}{2} \times \text{height}$$

  This is the formula for the area of a trapezium.

- **Understanding the Formula**

- The bases are the lengths of the parallel sides.

  The height is the perpendicular distance between the two bases.

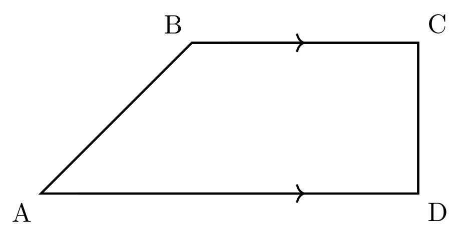
  The average of the bases gives us a ”typical length” for the trapezium, and when multiplied by the height, it gives the area, just like how you would calculate the area of a rectangle.

Key Takeaway

A trapezium is a four-sided polygon (quadrilateral) that has one pair of opposite sides that are parallel. These parallel sides are called the bases of the trapezium. The other two sides are not parallel and are called the legs.

Proterties of a trapezium.

- A trapezium looks like a bridge. Suggest other three real life examples of trapeziums.
- One pair of sides are parallel (they never meet).
- The height is the straight-up distance between the parallel sides.
- The angles inside add up to 360°.



Study Questions.

1. Define a trapezium in your own words.
2. Identify which of the following shapes are trapeziums giving reasons why:

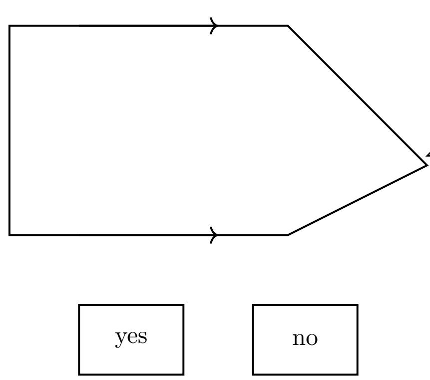
   a. A square

   b. A shape with one pair of parallel sides

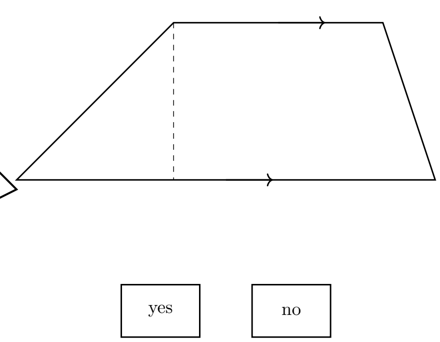
   c A rectangle

   - In the images provided below read the question and tick the appropriate answer.

     Is this polygon a trapezoid?

     Is this polygon a trapezoid?

3. Draw a trapezium and label its bases, legs, and height.

4. How many angles does a trapezium have? What is the sum of all interior angles?

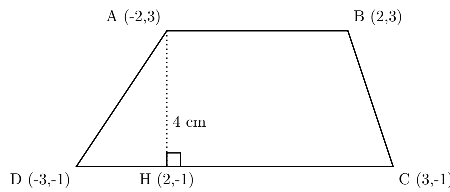
5. What is the difference between a trapezium and a parallelogram?

The area of a trapezium is given by the formula:

$$A = \frac{a+b}{2} \times h$$

where$a$ and b are the two parallel sides and h height.

Example 2.5.10 Calculate the area of the figure below.



Solution. The given figure is a trapezium with the following vertices:

- $A(-2, 3)$
- $B(2, 3)$
- $C(3, -1)$
- $D(-3, -1)$

  From the diagram, we observe:

- The two parallel bases are ABAB and DCDC.
- The height is given as 4 cm, which is the perpendicular distance between the parallel bases.

Step 1: Determine the lengths of the bases
Length of AB :

$$AB = |x_1 - x_2| = |2 - (-2| = |2 + 2| = 4\,\text{cm}$$

Length of DC :

$$DC = |x_1 - x_2| = |3 - (-3| = |3 + 3| = 6\,\text{cm}$$

Step 2: Use the trapezium area formula.
The area of a trapezium is given by:

$$\text{Area} = \frac{1}{2} \times (\text{Base}_1 + \text{Base}_2) \times \text{Height}$$

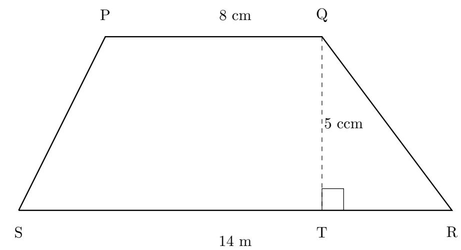
Substituting the values:

$$\begin{aligned} A =& \frac{1}{2} \times (4 + 6)\,\text{cm} \times 4\,\text{cm} \\ =& \frac{1}{2} \times 10\,\text{cm} \times 4\,\text{cm} \\ =& 20\,\text{cm}^2 \end{aligned}$$

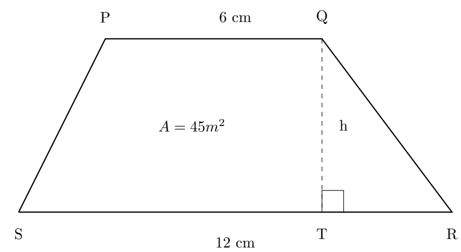
THe area of the trapezium is $20\,\text{cm}^2$ □

Exercise

1.A trapezium has a height of 5 cm, the top base is 8 cm and the bottom base is 14 cm. Find its area.



2. a) The bases of a trapezium are 12 cm and 6 cm and its area is 45 cm^2. Find the height.



b) Find the missing angle in a trapezium where three angles are 70°, 85° and 95°.

c) A trapezium has equal non-parallel sides (legs). What special type of trapezium is this?

d) The parallel sides of a trapezium are 20 m and 30 m and its height is 12 m. Find its area.

3. A trapezium-shaped farm has a shorter base of 150 m, a longer base of 300 m and a height of 200 m. Find the area of the farm.

4. A car window is shaped like a trapezium with a height of 50 cm, the top base is 60 cm and the bottom base is 80 cm. Find the area of the window.

5. A trapezoidal table has a top base of 1.2 m, a bottom base of 1.8 m and a height of 0.75 m. What is its surface area?

2.5.2.4 Area of a Kite

Activity 2.5.10 Constructing a Kite Using Geometry Tools. How to construct a kite accurately using a ruler, compass, and protractor.

- Materials needed:
- Graph paper or plain paper
- Ruler , Compass, protractor, pencil and eraser
- Follow this steps.

  Step 1: Drawing the diagnol.
- Draw a vertical line of length 10 cm (this will be the longer diagonal, d_1).
- Label the midpoint of this line as O.

  Step 2: Draw the Perpendicular Diagonal
- At O, use a protractor to draw a perpendicular line.

- Mark 4 cm on each side of O (total shorter diagonal $d_2 = 8cm$.

  Step 3: Mark the Kite's Vertices

- Label the four points where the diagonals intersect as A, B, C, and D.
- Connect **A** to **B**, **B** to **C**, **C** to **D** and **D** to **A**.

  Step 3:Check Properties

- Measure adjacent sides to ensure two pairs are equal.
- Verify opposite angles (one pair should be equal)
- Confirm diagonals are perpendicular.

Design your own kite patterns on graph paper and justify why their design would be aerodynamic and stable in the air.

Key Takeaway

A kite is a quadrilateral with two pairs of adjacent sides equal in length and one pair of opposite angles equal.

A kite can be analyzed using coordinate geometry when its vertices are given on a Cartesian plane.

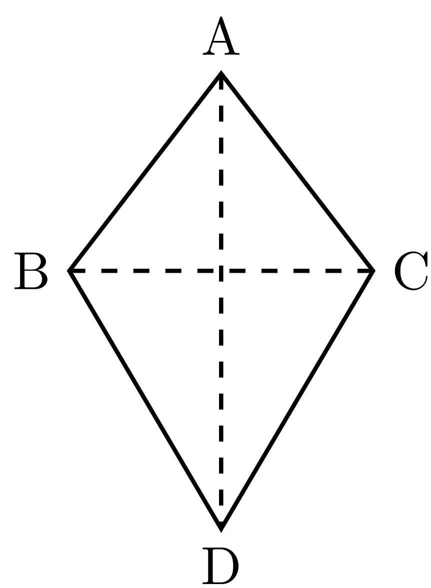
Since the diagonals are perpendicular, right-angled triangles are formed.

We can use trigonometric ratios (sine, cosine, tangent) to find missing angles and side lengths.

Finding Angles Using Trigonometry

If a kite has diagonals d_2 and d_2 we can find interior angles using:

$$\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}}$$
$$\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}}$$
$$\tan\theta = \frac{\text{opposite side}}{\text{adjacent side}}$$



If a kite has diagonals 14 cm and 10 cm, then each right-angled triangle within the kite has:

$$\text{Base} = \frac{d_1}{2} = \frac{14}{2}$$

$$= 7\,\text{cm}$$

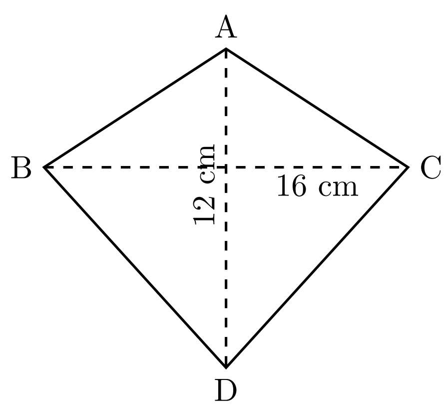
$$\text{Height} = \frac{d_2}{2} = \frac{10}{2}$$

$$= 5\,\text{cm}$$

The angle between the diagonal and a side can be found using:

$$\tan\theta = \frac{5}{7}$$
$$\theta = \tan^{-1} 0.7143$$
$$= 35.5°$$

Example 2.5.11 A kite whose has the diagonals are 16 cm and 12 cm.
a) Find the area of the a kite
b) If one pair of adjacent sides is 10 cm, find the perimeter of the kite.
c) Find the angles of the kite using trigonometry.



Solution. **Find the Area of the Kite** Let $d_1 = 16\,\text{cm}$ and $d_2 = 12\,\text{cm}$

$$\textbf{Area} = \frac{1}{2} \times d_1 \times d_2$$

substitute $d_1 = 16\,\text{cm}$ and $d_2 = 12\,\text{cm}$

$$\text{Area} = \frac{1}{2} \times 16\,\text{cm} \times 12\,\text{cm}$$
$$= \frac{192}{2}\,\text{cm}^2$$
$$= 96\,\text{cm}^2$$

(b) Find the Perimeter of the Kite. The diagonals bisect each other at right angles, so each half-diagonal forms a right-angled triangle. $\frac{d_1}{2} = 6$ cm and $\frac{d_2}{2} = 8$cm Using the Pythagoras Theorem:

$$a^2 + b^2 = c^2$$
$$= 6^2 + 8^2$$
$$c^2 = 36 + 64$$

$$\sqrt{c^2} = \sqrt{100}$$
$$c = 10\,\text{cm}$$

Since a kite has two pairs of equal sides, the perimeter is:

$$\begin{aligned} P &= 2(a+b) \\ &= 2(10+10)\,\text{cm} \\ &= 40\,\text{cm} \end{aligned}$$

(c) Finding the Angles Using Trigonometry. Using trigonometry in the right-angled triangle:

$$\begin{aligned} \tan\theta &= \frac{\text{Opposite}}{\text{adjacent}} \\ &= \frac{6}{8} \\ \theta &= \tan^{-1} 0.75 \\ \\ \theta &= 36.87^\circ \end{aligned}$$

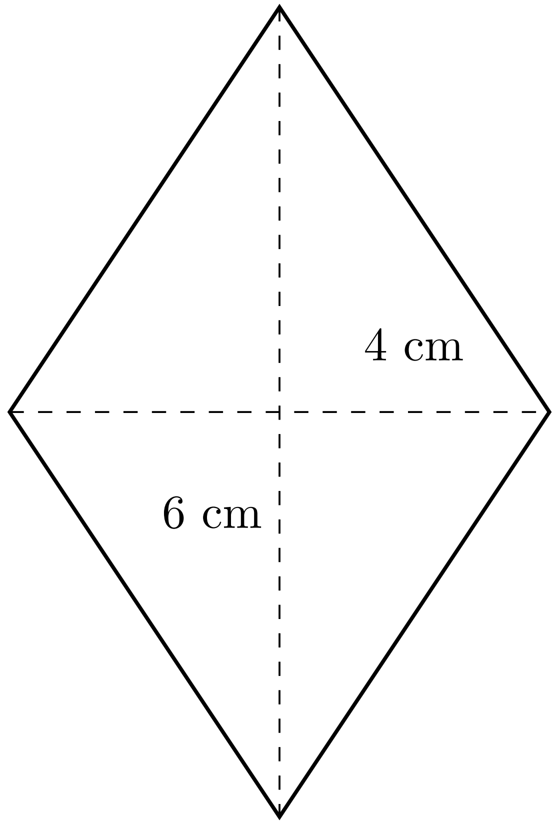
A kite is a quadrilateral, meaning it has four angles.
The sum of the interior angles of any quadrilateral is given by the formula:
Sum of Interior Angles=$(n-2) \times 180^\circ$
where $n = 4$*(since a kite has 4 sides)* $(4-2) \times 180^\circ = 2 \times 180^\circ = 360^\circ$
Since the kite is symmetric, the larger angle (α)is

$$\begin{aligned} &= 360^\circ - 2\theta \\ &= 360^\circ - 2(36.87)^\circ \\ &= 360^\circ - 73.74^\circ \\ 2(\alpha) &= 286.26^\circ \\ \alpha &= 143.13^\circ \end{aligned}$$

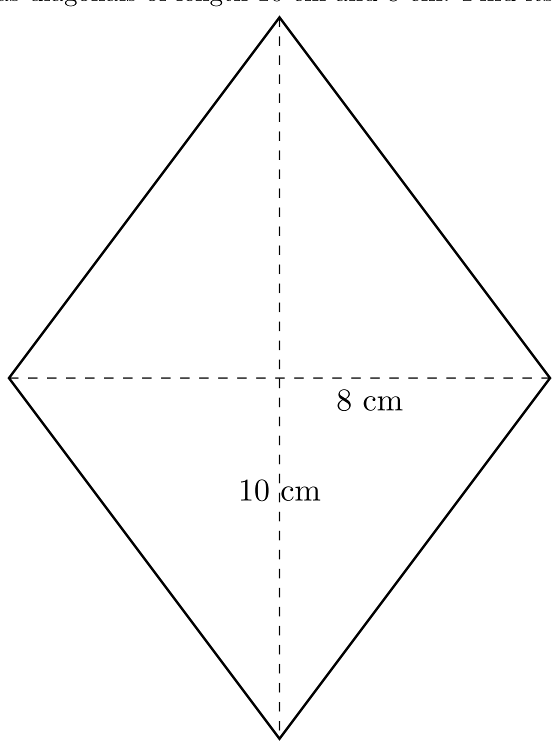
The angles of the kite are:143°, 143.13°, 36.87° and 36.87° □

Exercise

1. Find the area of the kite given below.



1. A kite has diagonals of length 10 cm and 8 cm. Find its area.



3. A playground has a kite-shaped design with diagonals measuring 15 meters and 12 meters. Find its area.

4. A rescue helicopter designates an emergency landing zone in the shape of a kite. The diagonals measure 60 m and 45 m. Calculate the available landing

space.

5. A playground has a kite-shaped design with diagonals measuring 15 meters and 12 meters. Find its area.A playground has a kite-shaped design with diagonals measuring 15 meters and 12 meters. Find its area.

6. Engineers are designing a kite-shaped solar panel with diagonals of 30 m and 18 m. What is the total solar-collecting area?

7. A relief team sets up a kite-shaped safe zone for disaster survivors. The diagonals measure 50 m and 40 m. What is the area available for the survivors?

2.5.3 Area of Regular Polygons

Activity 2.5.11 Materials Needed:

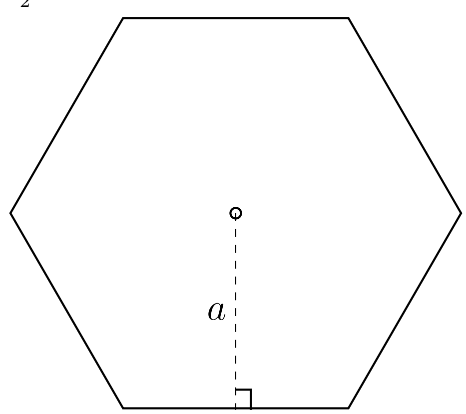
i.Compass,
ii. Ruler
ii. Protractor
iv. Calculator
v. Pencil and paper
vi. Colored pencils (optional)

- Draw a line from the center O perpendicular to a side (e.g., side BC). This line is the apothem (denoted a). Label the apothem clearly.

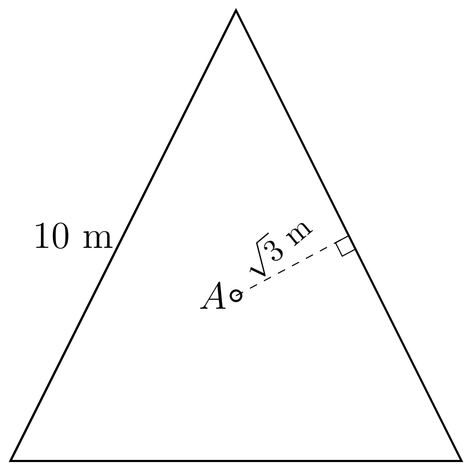
Key Takeaway

The **area** of a **regular polygon** can be found using this formula, where **P** is the **perimeter** and **a** is the apothem

The **apothem** is the distance from the center of a regular polygon to **a**side. Where **a** is the half $\frac{1}{2}$side.



You want to find the area of the triangle. You can see it has an apothem of $\sqrt{3}$ meters and a side length of 10 meters.



If you also find the perimeter, P , then you can use the formula A= $\frac{1}{2}\mathbf{P}a$ to find the area.

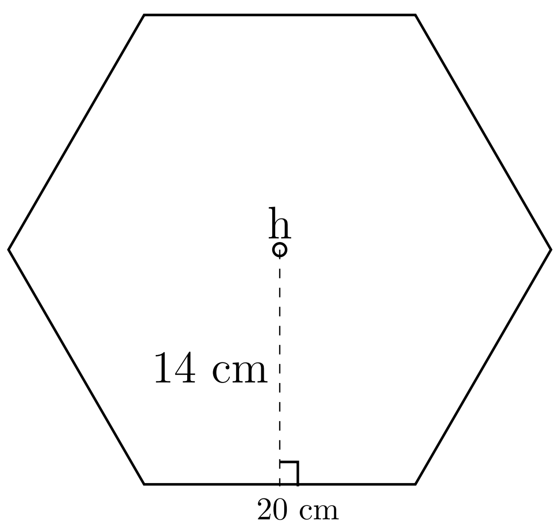
$$\mathbf{P} = \text{length of each side } \times a, \text{ where } a = h$$
$$P = 10\,\text{m} \times 3$$
$$= 30, \text{m}$$

Now, use the formula for the area of a regular polygon is :

$$A = \frac{1}{2} \times 30 \times \sqrt{3}$$
$$= 125.98\,\text{m}^2$$

Example 2.5.12 A regular hexagon with center h is shown below

Similarly you can use $A = \frac{1}{2} \times n \times S \times a$



Solution. Find the perimeter P, Then use the formula A= $\frac{1}{2}\mathbf{P}a$ to find the area.

$$\mathbf{P} = \text{length of each side } \times a$$
$$P = 20\,\text{cm} \times 6$$
$$= 120\,\text{cm}^2$$

Now, use the formula for the area of a regular polygon is :

$$A = \frac{1}{2} \times 120 \times 14\,\text{cm}$$
$$= 840\text{cm}^2$$

Where n = number of sides,S = length of each side and $a = \frac{S}{2}$

$$A = \frac{1}{2} n \times S \times a$$
$$= \frac{1}{2} \times 6 \times 20\,\text{cm} \times 14\,\text{cm}$$
$$= \frac{1,680}{2}\,\text{cm}^2$$
$$= 840\,\text{cm}^2$$

□

2.5.3.1 Area of Heptagon

A heptagon is a seven-sided polygon. It has seven edges and seven vertices.

Activity 2.5.12 Objective:

learners are expected to know how to construct a regular heptagon (7-sided polygon) using a compass, ruler, and protractor.

Materials Needed:

1. Materials Needed:

   Compass

   Ruler

   Protractor

   Pencil

   Eraser

   Graph paper (optional)

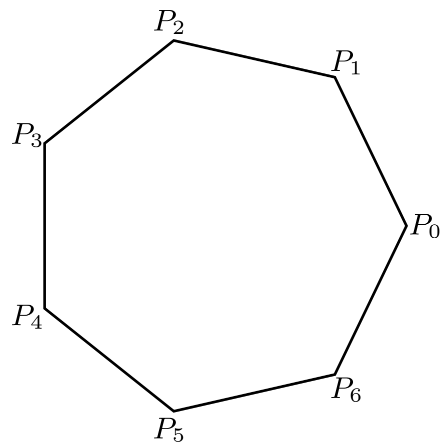
2. Step:1 Draw a Circle.
   - Place the compass pointer on the paper and draw a circle of any radius.
   - Mark the center (O) of the circle.

3. Step 2: Draw a Horizontal Diameter
   - Use the ruler to draw a straight line measuring 10cm through the center (O), creating a diameter (A 0 to P1).
   - Label the first point P1 on the circumference.

4. Step 3: Divide the Circle into Seven Equal Parts
   - Use a protractor to measure angles of $360° \div 7 = 51.43°$ from point P1.
   - Mark each 51.43° interval around the circle to get seven points.

5. Step 4: Connect the Points as shown below.

- Use the ruler to draw straight lines connecting the seven points in sequence.
- The heptagon is now complete!



Extended Activity.

Try drawing a heptagon using only a compass (without a protractor)

Shade the inside of the heptagon with different colors to make a pattern.

Find the sum of the interior angles of the heptagon. $(Hint : (n-2)\times 180°)$

Discussion Questions

1. What is the sum of all interior angles of a heptagon?

2. How do we calculate the measure of one interior angle of a regular heptagon?

3. Can you find a heptagon in real life (architecture, logos, etc.)?

Key Takeaway

Properties of a Regular Heptagon..

Sum of interior angles = $(n-2)\times 180°$. Where n is the number of sides.

Example for a Heptagon (7-sided polygon).

$$(n-2)\times 180°$$
$$(7-2)\times 180° = 900°$$

Sum of exterior angles of any **polygon(regular or irregular)** is always equl to 360° for both regular and irregular polygons.

Each Interior and Exterior Angle (Regular Polygon Only)

Each Interior Angle (for a regular polygon):

$$InteriorAngle = \frac{(n-2)\times 180°}{n}$$

Each Exterior Angle (for a regular polygon):

$$ExteriorAngle = \frac{360°}{n}$$

Example for a Regular Heptagon:

$$\textbf{Each Interior Angle} = \frac{900}{7}$$
$$= 128.57°$$

$$\textbf{Each Exterior Angle} = \frac{360}{7}$$

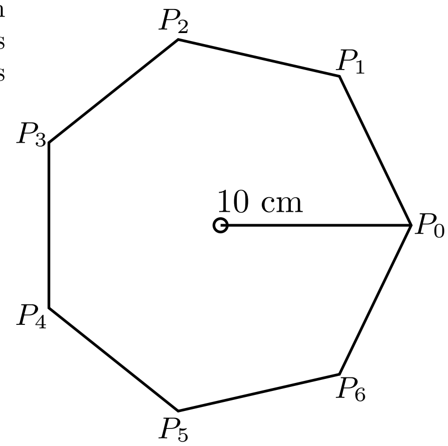
$$= 51.43°$$

Study Questions

1. What is the sum of all interior angles of a heptagon ?
2. How do we calculate the measure of one interior angle of a regular heptagon ?
3. Can you find heptagons in real life?

Example 2.5.13

A regular heptagon measures 10 cm as indicated in the figure below, find its area given the sum of it's interior angles is 900°?



Solution. There are 7 triangles since a heptagon has 7 sides. Area of Triangle $P_0OP_1 = \frac{1}{2}ab\sin\ 51.43°$

$$\begin{aligned}\triangle P_0OP_1 =& \frac{1}{2} \times 10 \times 10\ \ \sin 51.43° \\ =& \frac{1}{2} \times 100\,\text{cm}^2 \times\ \sin\ 51,43° \\ =& \frac{1}{2} \times 100\,\text{cm}^2 \times 0.7818 \\ =& 39.0923\,\text{cm}^2\end{aligned}$$

Therefore the total area

$$\begin{aligned}=& 7 \times 39.0923\,\text{cm}^2 \\ =& 273.65\,\text{cm}^2\end{aligned}$$

□

2.5.3.2 Area of a an Octagon

A regular octagon is an(8-sided) polygon with eight vertices (corners) and ten edges (sides).

Activity 2.5.13 Materials Needed:

Compass, Ruler, Protractor
Pencil, Eraser, Graph paper (optional)

Drawing a Regular Octagon (8-sided Polygon)

1. Draw a Circle:
Use a compass to draw a circle of your deired radius.
Mark the center (O).
2. Draw the First Diameter:

Use a ruler to draw a horizontal diameter (P0 to 95) passing through the center.

3. Divide the Circle into 8 Equal Parts.

Use a protractor to measure angles of 360° ÷ 8 = 45° from point P0.

Mark each 45° interval on the circle to get 8 points.

4. Connect the Points:

Use a ruler to connect the eight points in order to form the octagon.

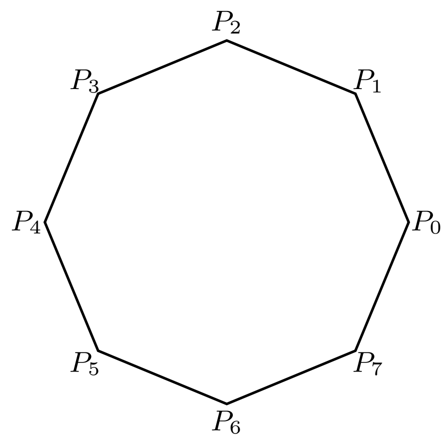
Discussion Questions

Study Questions

1. What is the sum of all interior angles of an octagon ?

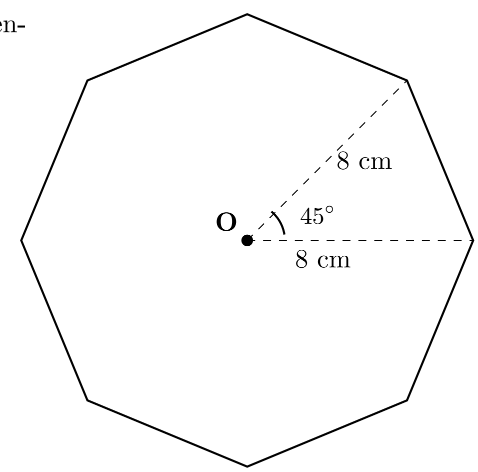
2, How do we calculate the measure of one interior angle of a regular octagon?

3, Have you ever seen octagon-shaped objects in real life?



Example 2.5.14

A regular octagon with O as its centre. If OA is 8 cm, find its area.



Solution. There are 8 triangles since an octagon has 8 sides. Area of Triangle $AOB = \frac{1}{2}ab\sin 45°$

$$\begin{aligned}\triangle AOB =& \frac{1}{2} \times 8 \times 8 \sin 45° \\ =& \frac{1}{2} \times 64\,\text{cm}^2 \times \sin 45° \\ =& \frac{1}{2} \times 64\,\text{cm}^2 \times 0.7071\end{aligned}$$

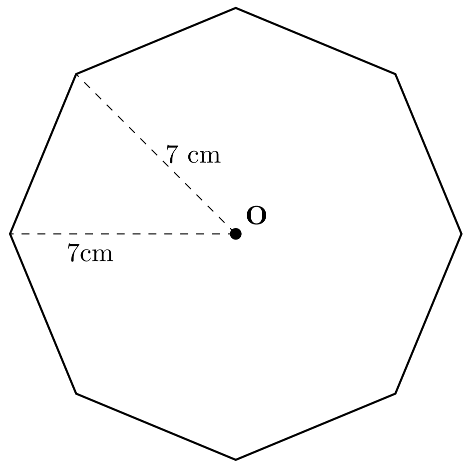
$$=22.6274\,\text{cm}^2$$

Therefore the total area

$$=8 \times 22.6274\,\text{cm}^2$$
$$=181.02\,\text{cm}^2$$

□

Example 2.5.15 The figure below represents a regular octagon with O as its centre. If OA is 7cm, find its area



Solution. There are 8 triangles since an octagon has 8 sides and the sum of interior angles of one triangle is 360°. Area of Triangle $AOB = \frac{1}{2}ab\sin 45°$.

The angle of $\triangle AOB = \frac{360}{8} = 45°$

$$\begin{aligned}\triangle AOB =& \frac{1}{2} \times 7 \times 7 \quad \sin 45° \\ =& \frac{1}{2} \times 49\,\text{cm}^2 \times \sin 45° \\ =& \frac{1}{2} \times 49\,\text{cm}^2 \times 0.7071 \\ =& 17.3241\,\text{cm}^2\end{aligned}$$

Therefore the total area

$$=8 \times 17.3241\,\text{cm}^2$$
$$=138.59\,\text{cm}^2$$

□

2.5.3.3 Area of a Nonagon

A nonagon is a nine-sided polygon with nine vertices (corners) and nine edges (sides).

Activity 2.5.14 Materials Needed:

Compass, Ruler, Protractor

Pencil, Eraser, Graph paper (optional)

Drawing a Regular nonagon (9-sided Polygon)

1. Draw a Circle:

Use a compass to draw a circle of your desired radius.
Mark the center (O).
2. Draw the First Diameter:
Use a ruler to draw a horizontal diameter (P0 to P6) passing through the center.
3. Divide the Circle into 9 Equal Parts.
Use a protractor to measure angles of $360° \div 9 = 40°$ from point P0.
Mark each 40° interval on the circle to get 9 points.
4. Connect the Points:
Study Questions

- What is the sum of all interior angles of an nonagon ?
- How do we calculate the measure of one interior angle of a regular nonagon?
- Have you ever seen nonagon-shaped objects in real life?

Example 2.5.16 Nonagon example.

One of the angles of a regular polygon is 40° and its sides are 15cm long. Sketch the polygon and then find the area of the polygon.

Solution.

From the definition of a polygon all interior angle add up to 360°.

To find the number of sides of a triangle we use the formula $\frac{360}{\text{Interior angle}} =$ Number of sides.

Where n is the number of sides.

$$\frac{360°}{\text{interior angle}} = \text{Number of sides}$$
$$\frac{360°}{40} = 9 \text{ sides}$$

Finding the area of the nonagon we use the formula:

Area of a tringle using sine rule $\times$the number of sides Where our ab is the radii.

$$\frac{1}{2}ab\sin 40° = \frac{1}{2} \times 15\text{cm} \times 15\text{cm} \times \sin 40°$$
$$= \frac{1}{2} \times 225 \times 0.6428 \text{ cm}^2$$

There are 9 triangles since a nonagon has 9 sides.
Therefore, total area is:

$$= 9 \times (\frac{1}{2} \times 225 \times 0.6428)$$
$$= 9 \times 225 \times 0.3214$$
$$= 73.6006 \text{ cm}^2$$

□

2.5.3.4 Area of a Decagon

A decagon is a ten-sided polygon with ten vertices (corners) and ten edges (sides).

Activity 2.5.15 Materials neede:

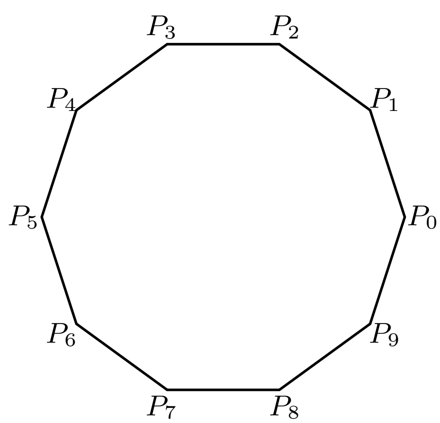
Compass, Ruler, Protractor,
Pencil, Eraser, Graph paper (optional)

Drawing a Regular Decagon (10-sided Polygon)

1. Draw a Circle:
Use a compass to draw a circle of your desired radius.
Mark the center (O).
2. Draw the First Diameter:
Use a ruler to draw a horizontal diameter (P0 to P6) passing through the center.
3. Divide the Circle into 10 Equal Parts.
Use a protractor to measure angles of 360° ÷ 10 = 36° from point A.
Mark each 36° interval on the circle to get 10 points.
4. Connect the Points:
Use a ruler to connect the ten points in order to form the decagon.

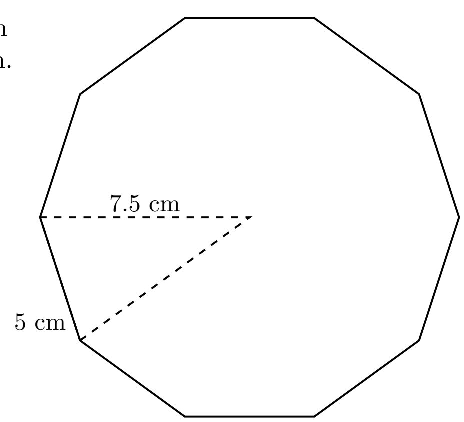
Study Questions

- What is the sum of all interior angles of an decagon ?
- How do we calculate the measure of one interior angle of a regular decagon?
- Can you find decagons in real life?
- What is the sum of all interior angles of a decagon?



Example 2.5.17

A regular decagon has an apothem a of 7.5cm and a side length S of5cm. Find its area.



Solution. Find the perimeter P, Then use the formula A= $\frac{1}{2}\mathbf{P}a$ to find the area.

$$\mathbf{P} = \text{length of each side } \times a$$
$$P = 5\,\text{cm} \times 10$$
$$= 50\,\text{cm}$$

Now, use the formula for the area of a regular polygon is :

$$\begin{aligned} A =& \frac{1}{2} \times 50 \times 7.5\,\text{cm} \\ =& \frac{375}{2}\,\text{cm}^2 \\ =& 187.5\,\text{cm}^2 \end{aligned}$$

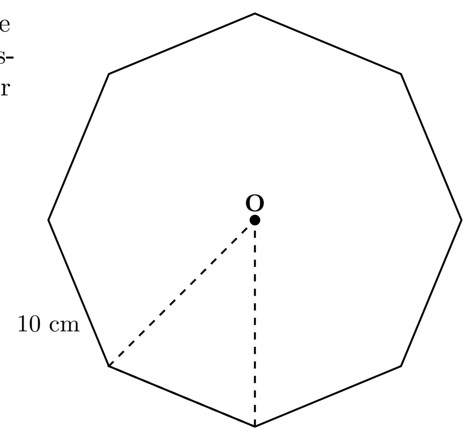
Where n = number of sides,S = length of each side and $a = \frac{S}{2}$

$$\begin{aligned} A =& \frac{1}{2} n \times S \times a \\ =& \frac{1}{2} \times 10 \times 5\,cm \times 7.5\,\text{cm} \\ =& \frac{375}{2}\,\text{cm}^2 \\ =& 187.5\,\text{cm}^2 \end{aligned}$$

□

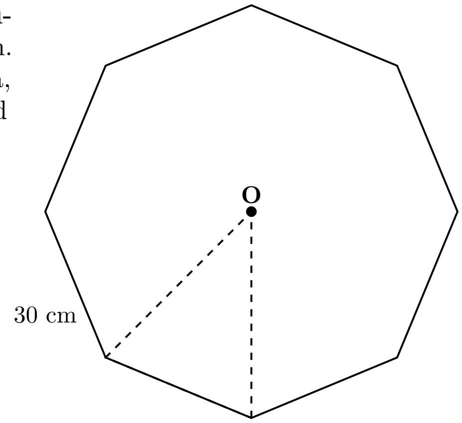
Exercise

1. A regular octagon has a side length of 10cm. Calculate its area using the formula for the area of a regular polygon.



2. A regular nonagon is inscribed in a circle of radius5cm. Find its area and perimeter, considering that all sides are equal and the internal angles can be derived using geometric properties.
3. The perimeter of a regular decagon is 50cm. Determine the length of each side and use it to calculate the area of the decagon.

4. A stop sign is shaped like a regular octagon with a side length of 30cm. If the material costs are based on area, determine the total surface area needed for manufacturing 100 signs.



5. A company designs floor tiles shaped like regular decagons with a side length of 12cm. If a room requires 50 tiles, calculate the total area covered by the tiles.

Checkpoint 2.5.18 An amusement park is the shape of a regular pentagon. Each side of the park measures 106 m and the height of the pentagon park is 40 m. Calculate the total area of the park.

The area of the park = ________ m^2

Answer. 10600

Solution. ***Given:*** • Side length = 106 m • Height of one triangle (from the center to the midpoint of a side) = 40 m

Area of pentagon = 5 × area of one triangle

Find the area of one triangle

Each triangle has: • Base = side of the pentagon = 106 m • Height = 40 m

$$\begin{aligned}\text{Area of one triangle} &= \frac{1}{2} \times \text{base} \times \text{height}\\ &= \frac{1}{2} \times 106 \times 40\\ &= 2120\,\text{m}^2\end{aligned}$$

Find the total area of the pentagon

$$\begin{aligned}\text{Total area} &= 5 \times \text{area of one triangle}\\ &= 5 \times 2120\\ &= 10600\,\text{m}^2\end{aligned}$$

Therefore, the total area of the pentagon-shaped park is 10600 m^2.

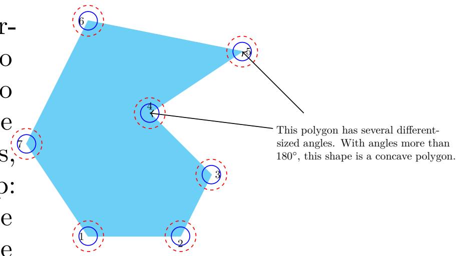
Checkpoint 2.5.19 An error occurred while processing this question.

2.5.4 Area of Irregular Polygons

Activity 2.5.16 Graph Paper (preferably colored)

Scissors
Glue or tape
Rulers
Graph paper (optional)
Measuring tape (optional)

A variety of irregular polygons (either printed or hand-drawn)



- work with a printed or drawn irregular polygon. The task is to divide the irregular polygon into smaller shapes whose areas are easier to calculate (like triangles, rectangles, or trapeziums). Tip: Use straight lines to cut along the diagonals or through the middle of the shape to create triangles or rectangles.

- For each smaller shape, ask students to measure the necessary dimensions:

  For triangles, they need the base and height.

  For rectangles, they need the length and width.

  For trapeziums, they need the lengths of the two parallel sides and the height.

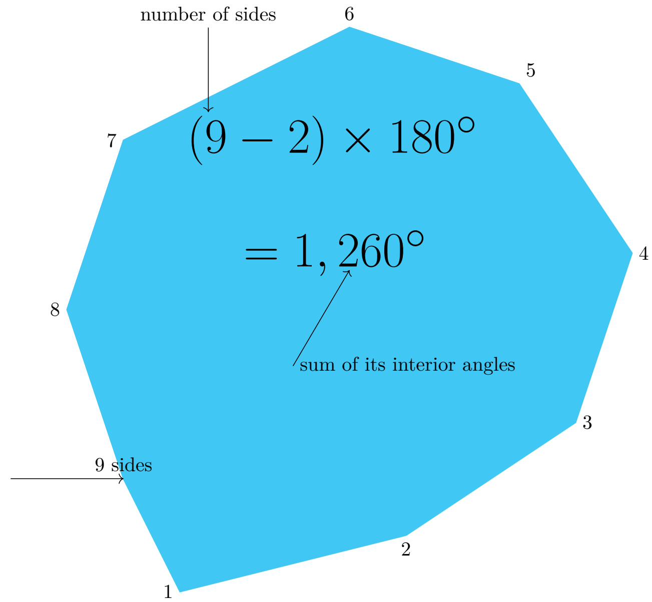
- Calculate the Area of Each Shape: Once the polygon is broken into smaller shapes, students can calculate the area of each shape using appropriate formulas

- Add the Areas: Once the area of each smaller shape has been calculated, students should add up all the areas to find the total area of the original irregular polygo

Regardless of shape, all polygons are made up of the same parts, sides, vertices, interior angles and exterior angles which may varry in size thus describing why we have irregular polygons versus regular polygons.

An irregular polgon has a set of atleast two sides or angles that are not the same. This heptagon has many different size angles , making it irregular.

The interior angles of an irregular nenagon (9 sides) add up to $1,260°$. Because angles are different sizes, individual angles cannot be found the sum of the interior angle.



Irregular nonagon
The interior angles of an irregular nonagon (9 sides) add up to 1,260°. Because the angles are different sizes, individual angles cannot be found from the sum of the interior angles.

Exercise

1. An irregular pentagon has the following side lengths: $5cm, 7cm, 6cm, 8cm and 4cm$. If its total area is estimated using triangulation, determine its approximate area.

2. A garden is shaped like an irregular hexagon with side lengths $4m, 6m, 5m, 7m, 8m and 3m$. Calculate its perimeter.

3. A farmer's land is shaped like an irregular quadrilateral with sides measuring $50m, 60m, 40m and 30m$. If the land is divided into two triangles for calculation, estimate its total area.

4. An office space has an irregular pentagonal shape with different side lengths and angles. The flooring cost is calculated based on the total area. If the room is divided into three triangles for estimation, find the approximate flooring cost given a rate of \$25 per square meter.

5. A city park is designed in the shape of an irregular hexagon with measured sides of $20m, 25m, 30m, 28m, 22m and 18m$. If the park's area is estimated by splitting it into smaller triangles, find the total area.

6. A regular decagon has a side length of $12cm$. Calculate its perimeter and area using the formula for the area of a regular polygon.

7. A large conference room has an octagonal shape with a side length of $9m$. If the flooring material costsKsh.30 per square meter, find the total flooring cost.

8. A heptagonal garden has side lengths of $5m, 7m, 6m, 8m, 9m, 6m, and 10m$. The owner wants to fence the garden. Calculate the total length of fencing required.

9. A nonagonal water tank has a radius of $5m$. If it is filled with water, determine the total volume assuming the depth is $4m$.

10. A decorative fountain is designed in the shape of a regular decagon with a side length of 15m. If the cost of tiling is \$40 per square meter, determine the total cost of tiling the fountain area.

11. A regular nonagon is inscribed in a circle of radius 7cm. Compute its side length and area.

12. A heptagonal plot of land has side lengths of $8m, 10m, 12m, 9m, 11m, 13m and 7m$. Find its perimeter. If the area is approximated by dividing it into triangles,

estimate its total area.

Technology 2.5.20 "We used to measure with rulers, now we measure with tools that reveal more than the eye can see. Technology is here, not to replace thinking but empower it." Kindly use the links below and explore in these interactive exercises.

Interactive triangles.

https://www.mathsisfun.com/geometry/triangles-interactive.html by Math is Fun

Interactive Quadrilaterals.

https://www.mathsisfun.com/geometry/quadrilaterals-interactive.html by Math is Fun

Interactive polygons.

https://www.mathsisfun.com/geometry/polygons-interactive.html by Math is Fun

2.6 Area of a Part of a Circle

In Grade 8, you learned how to find the area and arc length of a sector.

Now, you'll go a step further and discover how to find the area of different parts of a circle.

This includes:

- the area of an annulus (a ring shape),
- the area of a sector,
- the area of an annular sector (a slice of a ring),
- and the area where circles overlap.

2.6.1 Area of an Annulus

Activity 2.6.1 Work in groups

What you require: Circular objects (e.g., two different-sized cups, lids, or rings), ruler or measuring tape and pen and paper (or a calculator).

1. Look around your surroundings and find two circular objects that can fit inside each other (e.g., two different-sized bowls, two bottle caps, or two CDs).
2. Place the smaller object inside the larger one to visualize the annulus (ring shape).
3. Record the values.
   - Measure the radius of the larger circle (R) from its center to the edge.
   - Measure the radius of the smaller circle (r) in the same way.
4. Square both radii. and record your result.

   Subtract the squared radius of the smaller circle from the squared radius of the larger circle.

   Multiply the result by $\frac{22}{7}$ or 3.142
5. Discuss with your group how to calculate the area of an annulus.

6. Try this activity with different circular objects and compare your results.

Extended Activity

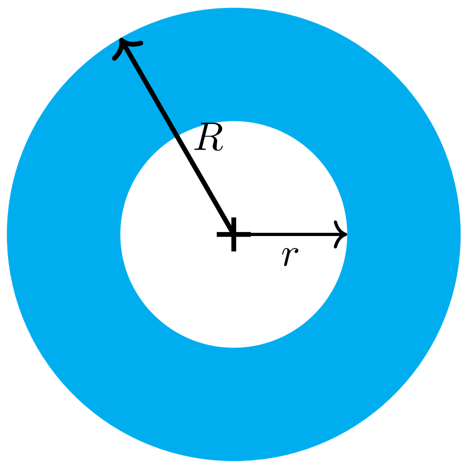
Activity 2.6.2 Individual work

Situation: Imagine a running track built around a circular field. The track has an inner boundary (smaller circle) and an outer boundary (larger circle). The track itself forms an annulus.

- A school is planning to paint the running track.
- The inner radius of the track is 30 meters, and the outer radius is 35 meters.
- The cost of painting is Ksh 5 per square meter
- Calculate:
  1. The area of the track
  2. The total cost of painting the track.

Key Takeaway

An **annulus** is the region between two concentric circles that share the same center but have different radii as shown below.



The area of an annulus (a ring-shaped object) is found by subtracting the area of the smaller, inner circle from the area of the larger, outer circle.

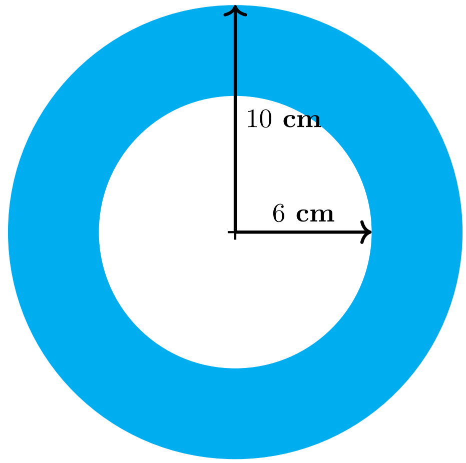
The formula is:

$$\begin{aligned} A_{\text{annulus}} =& A_{\text{outer circle}} - A_{\text{inner circle}} \\ =& \pi R^2 - \pi r^2 \\ =& \pi(R^2 - r^2) \end{aligned}$$

Where:

- R is the radius of the outer circle
- r is the radius of the inner circle

Example 2.6.1 Find the area of an the annulus drawn below;



Solution. $R = 10\,cm$ and $r = 6\,cm$

$$\begin{aligned} A =& \pi(R^2 - r^2) \\ =& \pi(10^2 - 6^2) \\ =& \pi(100 - 36) \\ =& 64\pi \\ =& 64 \times \frac{22}{7} \\ =& 201.06\,cm^2 \end{aligned}$$

□

Example 2.6.2 A wheel has an outer radius of $40\,cm$, and its inner hub has a radius of $10\,cm$. Find the area of the wheel's annular region.

Solution. The outer radius of the wheel $= 40\,cm$.

Inner hub radius $= 10\,cm$

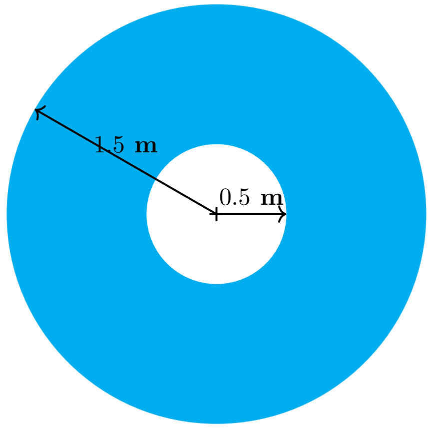
$$\begin{aligned} A =& \pi(R^2 - r^2) \\ =& \frac{22}{7}(40^2 - 10^2) \\ =& \frac{22}{7}(1\,600 - 100) \\ =& \frac{22}{7} \times 1\,500 \\ =& 4\,712.39\,cm^2 \end{aligned}$$

□

Exercises

1. A ring-shaped garden has an outer radius of 12 meters and an inner radius of 7 meters. Find the area of the garden.

2. A circular tabletop has a hole in the middle for an umbrella. The outer radius of the table is $1.5\,m$, and the hole has a radius of $0.5m$ as shown below.

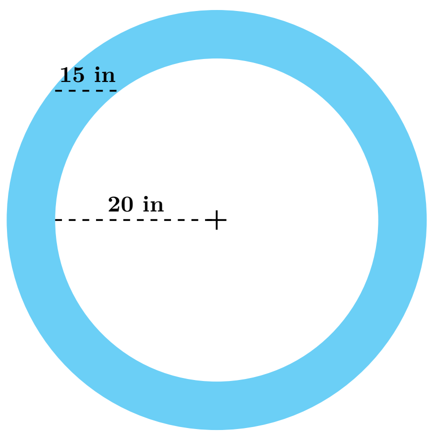


Find the area of the tabletop.

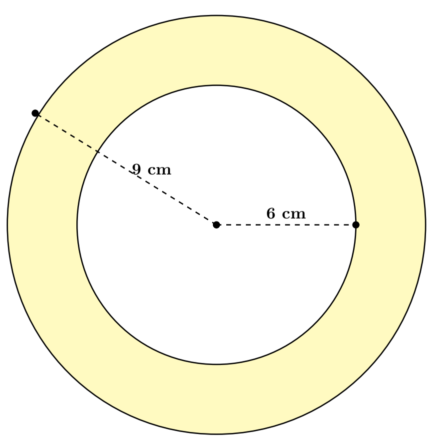
3. A circular swimming pool has an outer radius of 8 meters, and a smaller circular island is in the center with a radius of 2 meters. Find the area of the water surface.

4. A steel pipe has an outer diameter of 80 units and an inner diameter of 60 units, what is the area of the cross-section?

5. Find the area of the figure below; use $\pi = 3.142$



6. What is the area of the annulus shown; (Leave your answer to 3 significant figures).



Checkpoint 2.6.3 This question contains interactive elements.

Checkpoint 2.6.4 A wheel consists of a circular outer rim and a circular inner hub, both having the same centre. The outer radius of the wheel is 105 cm, and the radius of the inner hub is 63 cm. Using $\pi = 3.142$, determine the area of the wheel region.

The area of the wheel region =_______ cm^2

Answer. 22169.952

Solution. The outer radius of the wheel = 105 cm. Inner hub radius = 63 cm

Therefore,

$$\begin{aligned} A &= \pi(R^2 - r^2) \\ &= 3.142(105^2 - 63^2) \\ &= 3.142(11025 - 3969) \\ &= 3.12 \times 7056 \\ &= 22169.952\,\text{cm}^2 \end{aligned}$$

Therefore, The area of the wheel region = 22169.952 cm^2

2.6.2 Area of a Sector of a Circle

Activity 2.6.3 Work in groups

What you require:

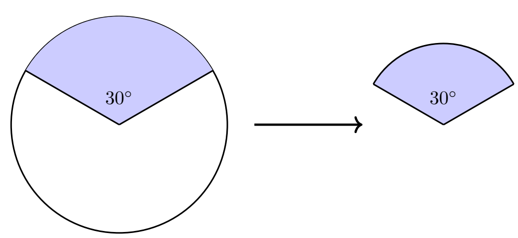
A graph paper and a razorblade or a pair of scissor

(i) Draw a circle of radius 7 cm on a graph paper.

(ii) Cut out the circle along its boundary.

(iii) Mark the centre of the circle.

(iv) Measure an angle of 30° at the centre and cut out as shown.



(v) Estimate the area by counting the number of squares enclosed by the arc and the two radii of the circle.

(vi) Express the angle of the sector (30°) as a fraction of the angle at the centre of the circle (360°).

(vii) Multiply the fraction obtained in (6) by the area of the circle.

(viii) Discuss and share the result with other groups.

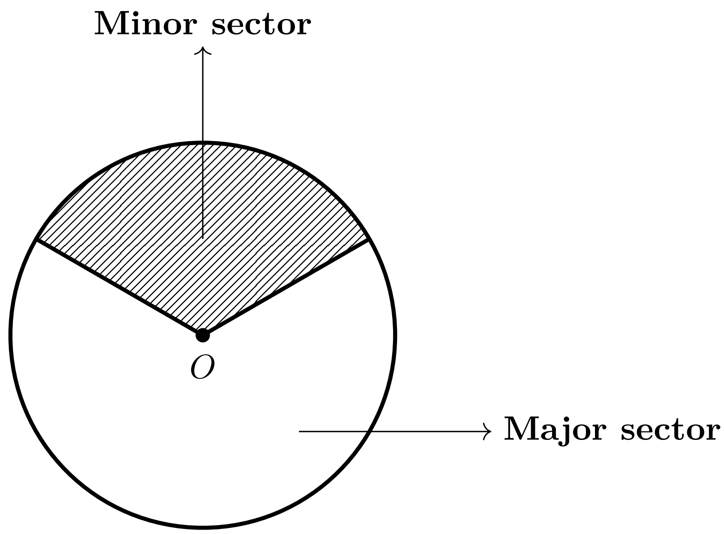
Key Takeaway

A **sector** is a region bounded by two radii and an arc.

Minor sector is one whose area is less than a half of the area of the circle.

Major sector is onewhose area is greater than a half of the area of the circle.

See the figure below;



The Area of a Sector

$$\text{Area of a Sector} = \frac{\theta}{360} \times \pi r^2$$

where:

- θ is in degrees,
- r is the radius of the circle,
- π 3.142 or $\frac{22}{7}$.

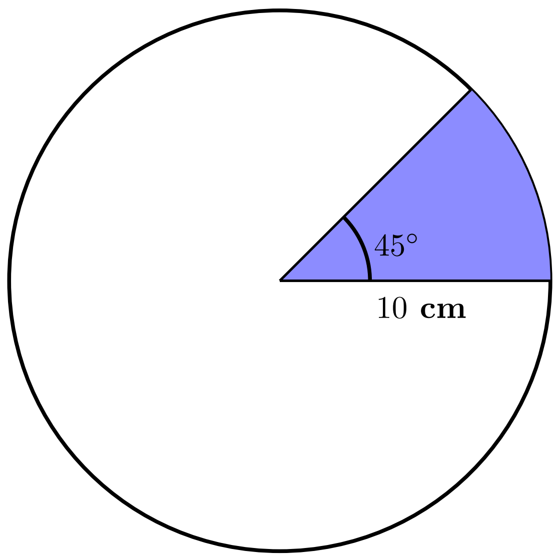
Example 2.6.5 Find the area of a sector of a circle of radius $7\,cm$ if the angle subtended at the centre is $90°$.

Solution. The values given are, $\theta = 90°$, $r = 7\,cm$
Area = $\frac{\theta}{360} \times \pi r^2$

$$\begin{aligned} \text{Area} =& \frac{90}{360} \times \frac{22}{7} \times (7^2) \\ =& \frac{1}{4} \times \frac{22}{7} \times 49 \\ =& \frac{1}{4} \times 22 \times 7 \\ =& 38.5\,cm^2 \end{aligned}$$

□

Example 2.6.6 Find the area of a sector of a circle shown below;(use $\pi = 3.142$)



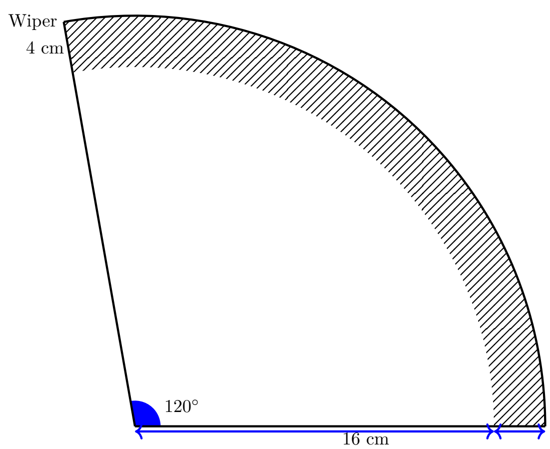
Solution. The values given are, $\theta = 45°$, $r = 10\,cm$

Area $= \frac{\theta}{360} \times \pi r^2$

$$\text{Area}\frac{45}{360} \times 3.142 \times (10^2)$$
$$=\frac{1}{8} \times 3.142 \times 100$$
$$=39.275\,cm^2$$

□

Example 2.6.7 The shaded region in the figure below shows the area swept out on a flat windscreen by a wiper. Calculate the area of this region.



Solution. The area of the rigion is goten by subtracting the **Area of the smaller sector** from **Area of the larger sector** .

Use Area $= \frac{\theta}{360} \times \pi r^2$

Area of the larger sector

$$R = 16\,cm + 4\,cm$$
$$= 20\,cm$$

$$\theta = 120^\circ$$

$$\begin{aligned} A =& \frac{120}{360} \times \frac{22}{7} \times 20^2 \\ =& \frac{1}{3} \times \frac{22}{7} \times 400 \\ =& 419.047619\, cm^2 \end{aligned}$$

Area of the smaller sector

$$r = 16\, cm$$

$$\theta = 120^\circ$$

$$\begin{aligned} A =& \frac{120}{360} \times \frac{22}{7} \times 16^2 \\ =& \frac{1}{3} \times \frac{22}{7} \times 256 \\ =& 268.19047\, cm^2 \end{aligned}$$

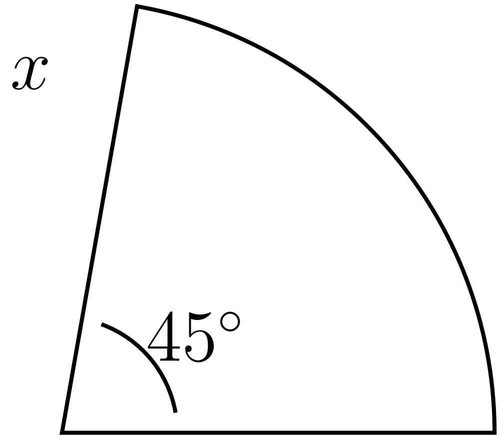
Therefore,

$$\begin{aligned} \text{Area of the region} =& \textbf{Area of the larger sector} - \textbf{Area of the smaller sector} \\ =& 419.047619\, cm^2 - 268.19047\, cm^2 \\ =& 150.85714\, cm^2 \end{aligned}$$

□

Exercises

1. A sector of a circle of radius $r$is subtended at the centre by an angle of θ. Calculate the area of the sector if:
   (a) $r = 10\, m, \quad \theta = 264^\circ$
   (b) $r = 8.4\, cm, \quad \theta = 40^\circ$
   (c) $r = 1.4\, cm, \quad \theta = 80^\circ$
2. The area of a sector of a circle is cm^2 . Find the radius of the circle if the angle subtended at the centre is 140°. (Take $\pi = \frac{22}{7}$)
3. A goat is tethered at the corner of a fenced rectangular grazing field. If the length of the rope is $21\, m$, what is its grazing area?
4. Shown below is a sector of a circle, with radius $x\, cm$



The area of the sector is $18\pi\, cm^2$
Find the length of x

5. A sector has an angle of $\frac{\pi}{3}$ radians and a radius of $8\, cm$. Find its area.

Checkpoint 2.6.8 This question contains interactive elements.

Checkpoint 2.6.9 This question contains interactive elements.

2.6.3 Area of an Annular Sector

Activity 2.6.4 Work in groups
What you require;

- Two circular paper cutouts (one smaller, one larger),
- Scissors,
- Protractor,
- Ruler,
- Colored markers.

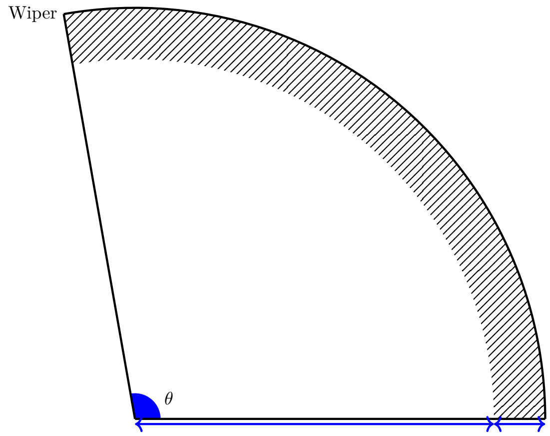
1. Take two circular cutouts of different sizes but with the same center.
2. Use a protractor to mark the same central angle θ on both circles.
3. Cut out the corresponding sectors from both circles.
4. Place the smaller sector on the larger one and observe the remaining shape.
5. Measure and calculate the area of each sector using the formula and compare with your actual cutout.
6. Discuss with other groups how to get the area of the figure you have formed.

Extended Activity

Activity 2.6.5 Work in groups
Materials Needed:

- A car wiper blade (or a picture of one) like the one drawn below;
- A protractor

- A ruler
- A notebook and calculator



1. Find the dimensions of the following;

   The length of the wiper blade.

   The pivot point to the base of the wiper blade.

   The angle θ is the angle through which the wiper moves.

2. Use the formula:

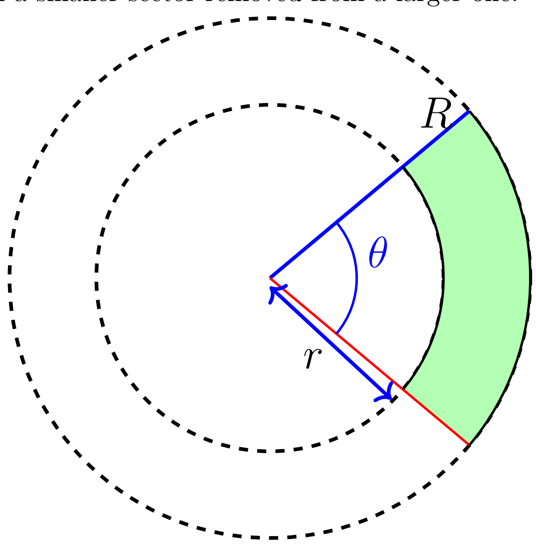
$$A = \frac{\theta}{360} \times \pi(R^2 - r^2)$$

   to calculate the area cleaned by the wiper.

3. Ask students in your group to observe whether the wiper covers all parts of the windshield equally.

Key Takeaway

An **annular sector** is the region enclosed between two concentric sectors of a circle with different radii but the same central angle. It is similar to a sector but with a smaller sector removed from a larger one.



Having the knowledge of the area of a sector and the area of an annulus it is very easy to identify the area of an annular sector.

Area of an annular sector

Area of an annular sector is:

$$A = \frac{\theta}{360} \times \pi(R^2 - r^2)$$

Example 2.6.10 A wind turbine blade sweeps through a central angle of 140°. The length of the blade is $50\,m$, and the inner radius (distance from the pivot to the base of the blade) is $10\,m$. Find the swept area.

Solution. The leargar radius $(R) = 50\,m$

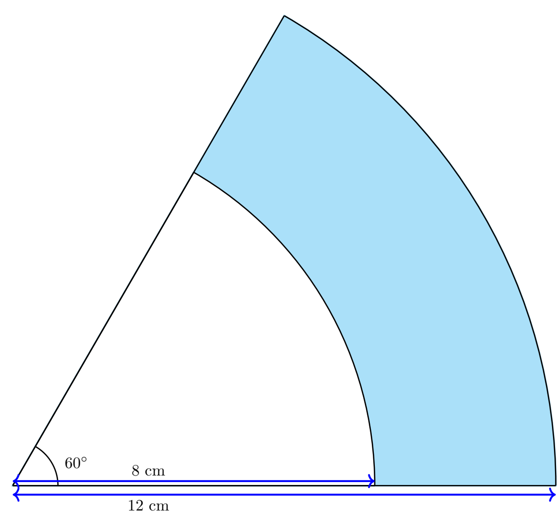
The inner radius $(r) = 10\,m$

Angle subtended $= 140°$

$$\begin{aligned} A =& \frac{140}{360} \times \pi(50^2 - 10^2) \\ =& \frac{7}{18} \times \frac{22}{7}(2\,500 - 100) \\ =& \frac{7}{18} \times \frac{22}{7} \times 2\,400 \\ =& 2933.333333 \\ & 2\,933.3\,m^2 \end{aligned}$$

The turbine sweeps an area of approximately $2\,933.3\,m^2$. □

Example 2.6.11 Find the area of the annular sector shown below. (Use $\pi = 3.142$)



Solution. $A = \frac{\theta}{360} \times \pi(R^2 - r^2)$

$$\begin{aligned} A =& \frac{60}{360} \times \pi(12^2 - 8^2) \\ =& \frac{60}{360} \times 3.142(144 - 64) \\ =& \frac{1}{6} \times 3.142 \times 80 \\ =& \frac{1}{3} \times 3.142 \times 40 \end{aligned}$$

$$=41.89333333$$
$$41.89\,cm^2$$

□

Exercise

1. A clock's minute hand moves 150° in 25 minutes. The minute hand is 15 cm long, and the inner radius is 5 cm Calculate the cleaned area.

2. A windshield wiper moves through 110°. The blade is 45 cm long, and the pivot distance is 15 cm. Calculate the cleaned area.

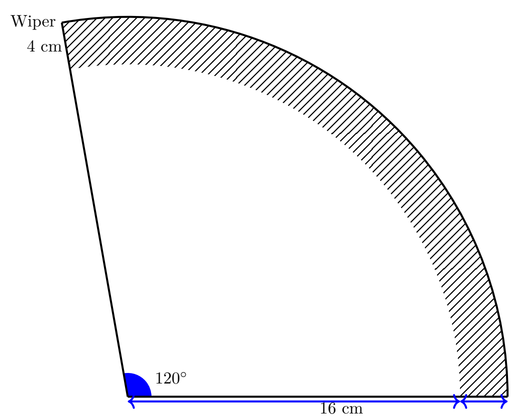
3. A car wiper has:

   - Outer radius = 40 cm
   - Inner radius $r = 10\,cm$
   - Central angle = 120°

   Find the area cleaned by the wiper.

4. A mechanical arm sweeps through 180°. The outer radius is 8 m, and the inner radius is 2 m. Determine the area covered.

5. The shaded region in the figure below shows the area swept out on a flat windscreen by a wiper. Calculate the area of this region.



Checkpoint 2.6.12 This question contains interactive elements.

Checkpoint 2.6.13 A mechanical arm sweeps through 120. The outer radius is 56 m and the inner radius is 21 m. Calculate the following areas:

1. The area of the smaller circle formed by the arm. _______ m^2

2. The area of the larger circle formed by the arm. _______ m^2

3. The area of the region swept by the mechanical arm. _______ m^2

Answer 1. 1386
Answer 2. 9856
Answer 3. 2823
Solution.

1. ***Area of the smaller circle (inner radius):***

$$
\begin{aligned}
A_{\text{inner}} &= \pi r^2 \\
&= \frac{22}{7} \times (21)^2 \\
&= 1386 \text{ m}^2
\end{aligned}
$$

2. ***Area of the larger circle (outer radius):***

$$
\begin{aligned}
A_{\text{outer}} &= \pi R^2 \\
&= \frac{22}{7} \times (56)^2 \\
&= 9856 \text{ m}^2
\end{aligned}
$$

3. ***Area of the region swept by the mechanical arm:***

$$
\begin{aligned}
A_{\text{swept}} &= \frac{\theta}{360}\pi(R^2 - r^2) \\
&= \frac{120}{360} \times \frac{22}{7} \times \left(56^2 - 21^2\right) \\
&= 2823 \text{ m}^2
\end{aligned}
$$

Alternatively, you can subtract the area of the inner circle from the area of the outer circle the multiply by $\frac{120}{360}$:

$$
\begin{aligned}
A_{\text{swept}} &= (9856 - 1386) \times \frac{120}{360} \\
&= \frac{8470}{3} \text{ m}^2 \\
&= 2823 \text{ m}^2
\end{aligned}
$$

2.6.4 Area of a Segment of a Circle

Activity 2.6.6 Work in groups

What you require;

- A compass,
- A ruler,
- A protractor,
- A scientific calculator,
- A worksheet with given radii and angles

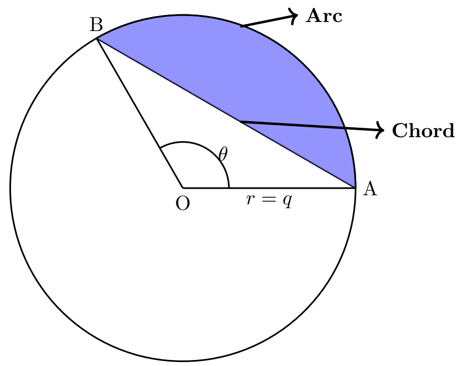
1. Use a compass to draw a circle of radius r.
2. Draw a chord across the circle using a ruler.
3. Use a protractor to measure the angle subtended at the center by the chord.

   Shade the segment formed

4. Find the area of the segment part and record your results.
5. Perform the same process for different chords and angles eg $70°, 90°, 120°, 150°$...
6. Discuss your work with other learners in your class.

Key Takeaway

A **segment** is the region of a circle bounded by a chord and an arc as shown in the figure below.



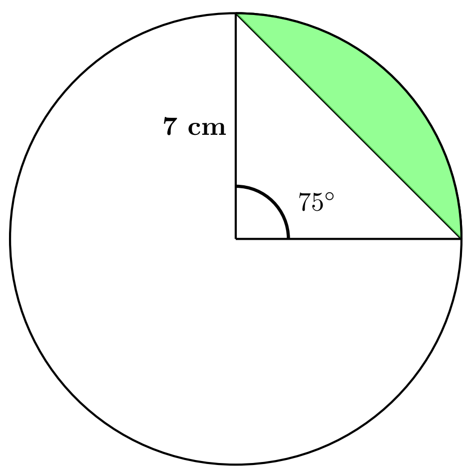
The shaded region is a segment of the circle with centre O and the radius q

Area of a segment of a circle.

The area of a segment is given by;

Area of a segment = Area of sector − Area of triangle

Example 2.6.14 A chord in a circle of radius $7\,cm$ subtends an angle of $75°$ at the center as shown below. Find the area of the segment.



Solution. To find the area of the segment, you the area of the triangle and subtract from the area of the sector.

Area of sector

$A = \frac{\theta}{360} \times \pi r^2$

$$\begin{aligned} A =& \frac{75°}{360} \times \frac{22}{7} \times 7^2 \\ =& \frac{5}{24} \times \frac{22}{7} \times 49 \\ =& \frac{385}{12} \end{aligned}$$

$$=32.0833\, cm^2$$

Area of a triangle
$A = \frac{1}{2}absin\,\theta$
Where $a = 7\,cm$ and $b = 7\,cm$

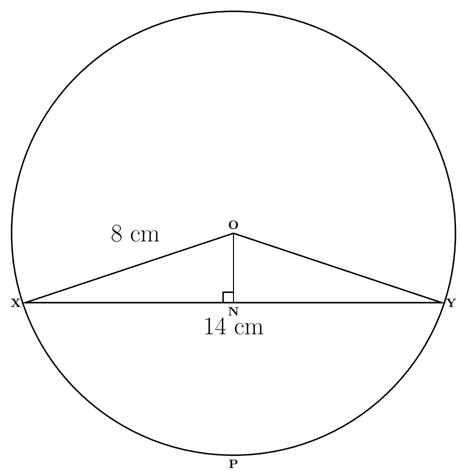
$$\begin{aligned} A =& \\ =&\frac{1}{2} \times 7 \times 7 \times \times sin\,75° \\ =&\frac{1}{2} \times 49 \times sin\,75° \\ =&23.6652\,cm^2 \end{aligned}$$

Therefor,

$$\begin{aligned} A =&32.0833\,cm^2 - 23.6652\,cm^2 \\ =&8.4181\,cm^2 \end{aligned}$$

□

Example 2.6.15 A chord XY of length 14 cm is drawn in a circle with centre O and radius 8cm, as in the figure below



Calculate:

(a) the distance ON.

(b) the area of the sector OXPY.

(c) the area of triangle OXY

(d) the area of the minor segment.

(e) the area of the major segment.

Solution. Given a circle with center O, radius 8 cm, and a chord XY of length 14 cm , we find the required values step by step.

(a) Finding the Distance ON

Since ON is the perpendicular bisector of XY, we use the right triangle ONX where:

$$\begin{aligned}\text{OX} =& 8\,cm(\text{radius})\\ \text{NX} =& \frac{\text{XY}}{2}\\ =& \frac{12}{2}\\ =& 7\,cm\end{aligned}$$

Using Pythagoras' Theorem in △ ONX:

$$\begin{aligned}\text{ON}^2 + \text{NX}^2 =& \text{OX}^2\\ \text{ON}^2 + 7^2 =& 8^2\\ \text{ON}^2 + 49 =& 64\\ \text{ON}^2 =& 64 - 49\\ \text{ON}^2 =& 15\\ \text{ON} =& \sqrt{15}\\ & 3.87\,cm\end{aligned}$$

(b) Finding the area of sector OXPY.

Finding the ∠ XOY which is given by:

$$\begin{aligned}cos\,\theta =& (\frac{\text{XN}}{\text{XO}})\\ =& \frac{7}{8}\\ =& 0.875\\ \theta =& cos^{-1}(0.875)\\ =& 28.9550°\end{aligned}$$

therefor, $\angle$XOY = 28.9550° × 2

$$= 57.9100°$$

The area of a sector is:

$$\begin{aligned}A =& \frac{\theta}{360} \times \pi r^2\\ =& \frac{57.9100}{360} \times \pi 8^2\\ =& \frac{57.9100}{360} \times \frac{22}{7} \times 64\\ =& 32.3561\,cm^2\end{aligned}$$

Therefore, area of sector OXPY = $32.3561\,cm^2$

(c) Finding the area of triangle OXY

$$\begin{aligned}\text{Area } =& \frac{1}{2} \times \text{base} \times \text{height}\\ =& \frac{1}{2} \times 14 \times 3.87\, cm\\ =& 7 \times 3.87\, cm\\ =& 27.09\, cm^2\end{aligned}$$

(d) Finding the area of the minor segment.

Area of minor segment = Area of sector − Area of△OXY

$$\begin{aligned}\text{Area of minor segment } =& \\ =& 32.3561\, cm^2 - 27.09\, cm^2\\ =& 5.2661\, cm^2\end{aligned}$$

(e) Finding the area of the major segment

Area of major segment = Total circle area−Area of minor segment

Area of a circle.

$$\begin{aligned}A =& \pi r^2\\ =& \frac{22}{7} \times 8^2\\ =& \frac{22}{7} \times 64\\ =& 201.1429\, cm^2\end{aligned}$$

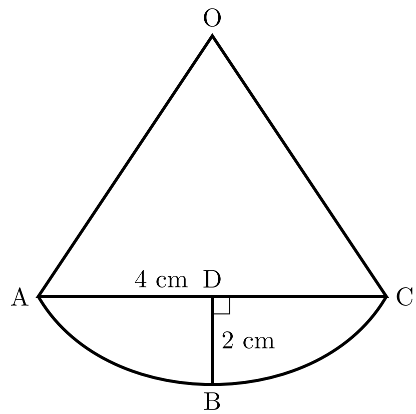
Therefore, Area of major segment is

$$\begin{aligned}=& 201.1429\, cm^2 - 5.2661\, cm^2\\ =& 195.8768\, cm^2\end{aligned}$$

□

Exercises

1. In the figure below, ADC is a chord of a circle with centre O passing through A, B and C. BD is a perpendicular bisector of AC. AD = 4 cm and BD = 2 cm.



Calculate:

(a) the radius OA of the circle.

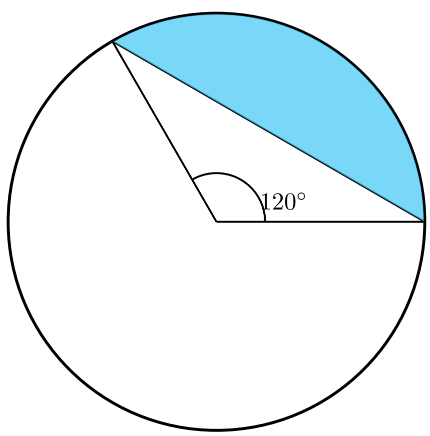
(b) the area of the sector OABC.

(c) the area of the segment ABCD

2. A circular table has a radius of 15 cm. A slice of cake is cut out, forming a 75° segment. Find the area of the cake slice not covered by the straight cut

3. A wheel of a car has a radius of 30 cm. A mudguard covers a 60° segment of the wheel. Find the area of the covered segment.

4. A circular park has a radius of 20 meters. A walking path cuts across the park, forming a chord that subtends an angle of 120° at the center as shown below.



5. A chord XY subtends an angle of 120° at the centre of a circle of radius 13 cm. Calculate the area of the minor segment.

Checkpoint 2.6.16 This question contains interactive elements.

Checkpoint 2.6.17 This question contains interactive elements.

2.6.5 Area of Common Region between two Intersecting Circles

Activity 2.6.7 work in groups

What you require; Compass, ruler, pencil, graph paper and Calculator.

1. Use a compass to draw a circle with a radius of $5\,cm$ centered at point O.
2. From a point $2\,cm$ to the right of O, draw another circle with the same radius ($5cm$).

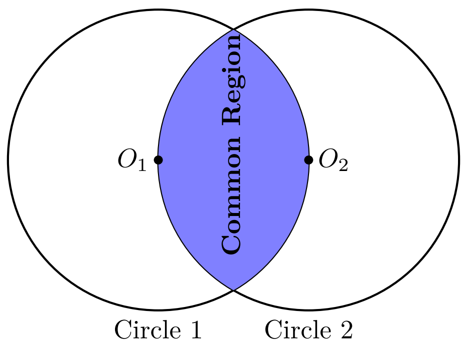
   This should create an overlapping region.
3. Name the intersection points of the two circles as P and Q.
4. Lightly shade the overlapping region between the two circles.
5. Finding the area of the common region.

   How does the distance between centers affect the common area?
6. Discuss your result with other learners in class.

Key Takeaway

The **common region between two intersecting circles** refers to the overlapping area shared by both circles.

It is formed when two circles of different or equal radii intersect at two distinct points as shown below.

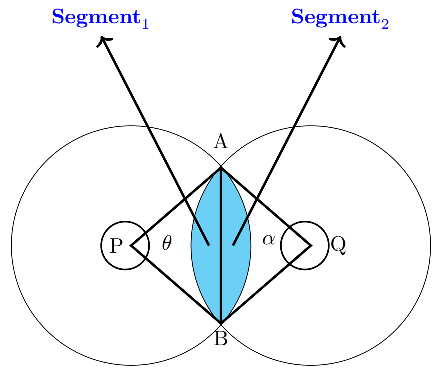


The area of the common region can be found by:

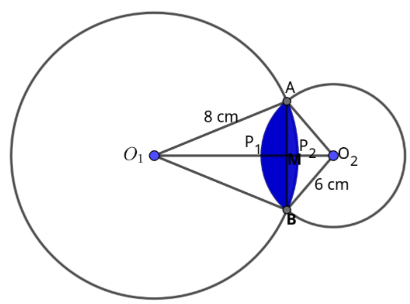
1. Calculating the area of the two circular segments formed by the chord joining the intersection points.
2. Sum the areas of the two segments.

$$A = A_{\text{segment 1}} + A_{\text{segment 2}}$$

The figure below shows the two segments.



Example 2.6.18 The figure below shows two circles of radii $8\,cm$ and $6\,cm$ with centres O_1 and O_2 respectively. The circles intersect at points A and B. The lines O_1O_2 and AB are perpendicular to each other. If the common chord AB is $9\,cm$, calculate the area of the shaded region.



Solution. From $\triangle AO_1M$;

$$\begin{aligned} O_1M =& \sqrt{8^2 - 4.5^2} \\ =& \sqrt{43.75} \\ =& 6.14\,cm \end{aligned}$$

From $\triangle AO_2M$;

$$\begin{aligned} O_2M =& \sqrt{6^2 - 4.5^2} \\ =& \sqrt{15.75} \\ =& 3.969\,cm \end{aligned}$$

The area of the shaded region is the sum of the areas of segments AP_1B and AP_2B. Area of segment AP_1B = area of sector C_2AP_1B–area of $\triangle O_2AB$.

Using trigonometry,

$$\begin{aligned} \angle AO_2M =& \frac{AM}{AO_2} \\ =& \frac{4.5}{6} \\ =& 0.75 \\ \angle =& sin^{-1}(0.75) \\ =& 48.59^\circ \end{aligned}$$

$$\begin{aligned}\angle AO_2B =& 2\angle AO_2M\\ =& 2 \times 48.59^\circ\\ =& 97.18^\circ\end{aligned}$$

Area of segment AP_1B
Area sector.

$$\begin{aligned}=&\frac{97.18}{360} \times 3.142 \times 6^2\\ =& 30.53\end{aligned}$$

Area of triangle.

$$\begin{aligned}=&\frac{1}{2} \times 9 \times 3.969\\ =& 17.86\end{aligned}$$

Therefore,

$$\begin{aligned}\text{Area of segment} AP_1B =& 30.53 - 17.86\\ =& 12.67\, cm^2\end{aligned}$$

Area of segment AP_2B = area of sector 0_1AP_2B–area of $\triangle O_1AB$.
Using trigonometry,

$$\begin{aligned}\angle AO_1M =& \frac{AM}{AO_1}\\ =& \frac{4.5}{8}\\ =& 0.5625\\ \angle =& sin^{-1}\,(0.5625)\\ =& 34.23^\circ\end{aligned}$$

$$\begin{aligned}\angle AO_1B =& 2\angle AO_1M\\ =& 2 \times 34.23^\circ\\ =& 68.46^\circ\end{aligned}$$

Area of segment AP_2B
Area sector.

$$\begin{aligned}=&\frac{68.46}{360} \times 3.142 \times 8^2\\ =& 38.24\end{aligned}$$

Area of triangle.

$$=\frac{1}{2} \times 9 \times 6.614$$

$$=29.76$$

Therefore,

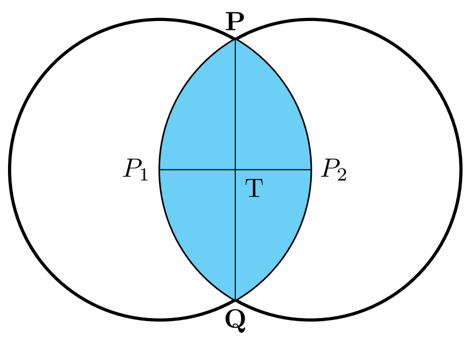
$$\text{Area of segment}AP_2B = 38.24 - 29.76$$
$$= 8.48\,cm^2$$

Therefore, area of the shaded region is given by; Area of segmentAP_1B + Area of segmentAP_2B

$$=12.67\,cm^2 + 8.48\,cm^2$$
$$=21.15\,cm^2$$

□

Example 2.6.19 The figure below shows two intersecting circles of radius $10\,cm$ each with centre P_1 and P_2. The length of P_1 and P_2 is $6\,cm$. (Take $\pi = 3.142$)



Find:

(a) the length of the common chord **PQ**.

(b) the area common to the two circles.

Solution. We are given:

- Two intersecting circles with radius $r = 10, cm.$.
- Distance between the centers: $P_1P_2 = 6\,cm$.
- The common chord PQ is perpendicular to P_1P_2 at T (the midpoint of PQ).

(a) Finding the Length of the Common Chord PQ

Applying the Right-angled triangle property

Since the chord PQ is perpendicular to the line joining the centers, we can analyze the right-angled triangle P_1TP.

Using the Pythagorean theorem in $\triangle P_1TP$;

$$P_1P^2 = P_1T^2 + PT^2$$

Where:

- $P_1P = 10\,cm$,
- $P_1T = \frac{P_1P_2}{2} = \frac{6}{2} = 3\,cm$.

Substituting values:

$$\begin{aligned} 10^2 =& 3^2 + PT^2 \\ 100 =& 9 + PT^2 \\ PT^2 =& 91 \\ PT =& 91 \\ =& \sqrt{91} \\ =& 9.54\,cm \end{aligned}$$

Since PQ $= 2PT$

$$\begin{aligned} \text{PQ} =& 2 \times 9.54\,cm \\ =& 19.08\,cm \end{aligned}$$

(b) Finding the common area between the two circle.

The common area consists of two identical circular segments, each subtended by the central angle 2θ at P_1 or P_2.

Finding θ

$$\begin{aligned} cos\,\theta =& \frac{P_1T}{P_1P} \\ =& \frac{3}{10} \\ =& 0.3 \\ \theta =& cos^{-1}\,0.3 \\ =& 72.54° \end{aligned}$$

$\angle P_1 = \angle P_2 = 2 \times 72.54°$

$$= 145.08°$$

Finding the segmet of P_1 and P_1. Since the segment are the same, it implies that area of segment $P_1 = P_2$.

Therefore,

Area of segment = area of sector − area of triangle

Area of sector

$$\begin{aligned} A =& \frac{145.08°}{360} \times 3.142 \times 10^2 \\ =& 126.62\,cm^2 \end{aligned}$$

Area of triangle

$$A = \frac{1}{2} \times 10 \times 10 \times sin\,145.08$$
$$= \frac{1}{2} \times 10^2 \times sin\,145.08$$
$$= 28.62\,cm^2$$

Area of segment $= 126.62\,cm^2 - 28.62\,cm^2$

$$= 98.00\,cm^2$$

Therefore,

Area of the common region $= 2 \times 98.00\,cm^2$

$$196.00\,cm^2$$

□

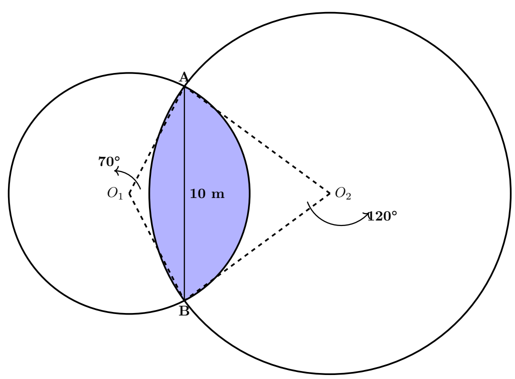
Exercises

(Take $\pi = 3.142$)

1. If two circles of radius r overlap such that their centers are at a distance $0.5\,r$, express the overlapping area in terms of r.

2. Two circular traffic islands of radius 10 meters overlap so that the centers are 12 meters apart. The angle subtended at the center of each circle by the chord of intersection is 120°. Find the area of the overlapping region.

3. Find the area of the figure below;



4. In the Olympic symbol, circles of radius $5\,cm$ overlap, forming intersections. If the central angle corresponding to the common region is 90°, find the area of the intersection.

5. During a rare planetary alignment, two planets appear as overlapping circles in the sky. If both have an apparent radius of $5000\,km$, and their centers are $6000\,km$ apart, with a central angle of 145°, find the overlapping shadow area visible from Earth.

Technology Integration: Exploring Areas of Part of a Circle

To deepen your understanding of how to find the area of different parts of a circle, explore the following interactive and insightful resources:

1. YouTube – Shifting Grades Online School

   This video lesson offers a visual and step-by-step explanation on how to find the area of parts of a circle. It's ideal for learners who prefer guided instruction with examples.

   https://youtu.be/r5Pxjp-uZfA

2. YouTube – Anil Kumar

   Learn about circle sectors and other related topics with this clear and concise tutorial. Great for quick revision or reinforcing concepts.

   https://youtu.be/Qd4iqL3SMkY

3. BBC Bitesize – Guide on Circles

   This interactive webpage from BBC Bitesize explains how to calculate the area of parts of a circle using helpful visuals and explanations.

   https://www.bbc.co.uk/bitesize/guides/z9hsrdm/revision/4

4. GeoGebra – Interactive Geometry Tool

   Use GeoGebra to explore and manipulate circle diagrams in real time. You can adjust radii and angles to see how the area of a sector or annulus changes perfect for visual learning and experimentation.

   https://www.geogebra.org/

Checkpoint 2.6.20 This question contains interactive elements.

Checkpoint 2.6.21 This question contains interactive elements.

2.7 Surface Area and Volume of Solids

Surface Area.

Activity 2.7.1 Materials Needed:

1. Solids made of cardboard or plastic (cube, rectangular prism, cylinder, cone, pyramid, sphere)

2. Grid paper or plain paper

3. Scissors, rulers, tape, glue and String (for measuring curved edges like the circumference of a circle)

4. Pre-made nets of solids (optional)

- Worksheet to record observations and answers
- Choose one solid object (e.g., cube, cone, cylinder).
- Create a net for their object (either by unfolding a model or using printed templates).
- Trace the faces onto grid paper or measure them using a ruler or string.
- Calculate the area of each face using appropriate formulas.
- Add up all face areas to find the total surface area.
- For cylinders and cones, use a string to measure the curved part.

Extended Activity

Wrap or cover real items (e.g., a cereal box, soda can) using calculated surface area. Design a custom box or can with a specific surface area for packaging a product.

Study Questions

1. How did your net help you find the surface area?
2. What would happen if your object was twice as big — would surface area double?
3. Derive and apply surface area formulas using practical reasoning and measurement.

Key Takeaway

Surface area is the total area of the exposed or outer surfaces of a prism.

A right prism is a geometric solid that has a polygon as its base and vertical faces perpendicular to the base. The base and top surface are the same shape and size. It is called a "right" prism because the angles between the base and faces are right angles.

The name given to the solid that is unfolded this way is called a net. When a prism is unfolded into a net, we can clearly see each of its faces.

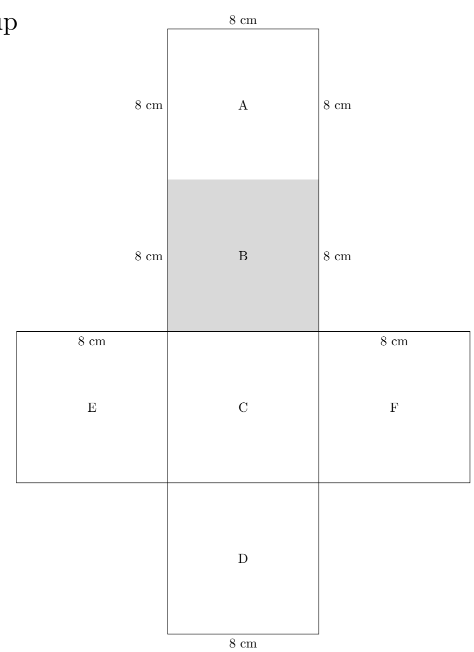
We can thereby clearly calculate the surface area by finding the area of each faces and add them all together to get the surface area of the Prism.

For example, If we are given a cylinder, the top and bottom faces are circles and the curved area is like a rectangle. So when finding it's surface area we find the area of the two circles and the area of the rectangle then we add all these areas all together to compute the cylinder's surface area.

2.7.1 Surface area of Prisms

Activity 2.7.2

A cube is unfolded to a net made up of 6 identical squares .



Materials needed.

- Paper Folds of multiple cubes.
- Rulers

- Glue

  Measure the side length (s) of one smaller square on a face.

  Count how many squares make up one face for example, a 3x3 or 4x4

  Make an entire cuboid.

  Count how many squares make up the cuboid.

- Discuss in your groups what will happen if the surface area doubles in size.

  Why do bigger cubes have more surface area?

Key Takeaway

A prism is a Geometric object with two identical, parallel bases and straight sides connecting them.

Examples of right prisms are cylinders, rectangular prisms, cubes and triangular prisms.

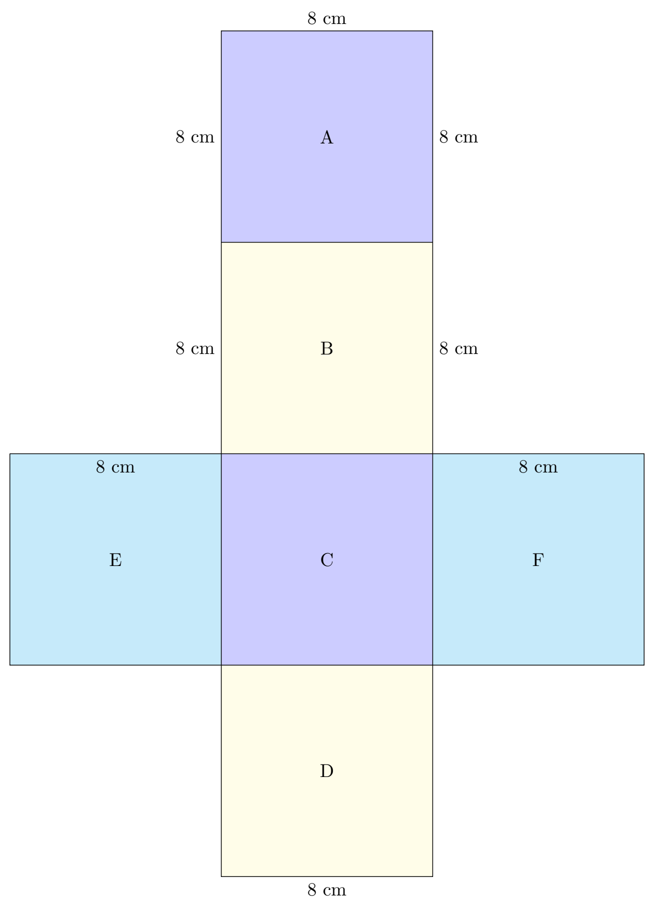
Types of Prisms: Rectangular and cube prisms, Triangular Prisms and Cylinders.

Surface Area of a cube

Observe the cuboid below whose side is 8cm.

Example 2.7.1 (a). Work out the surface area of the cube whose side is 8cm.

Solution.



(b).The surface area of the cube.

The area of one face.

$$= 8\,\text{cm} \times 8\,\text{cm}$$
$$= 64\,\text{cm}^2$$

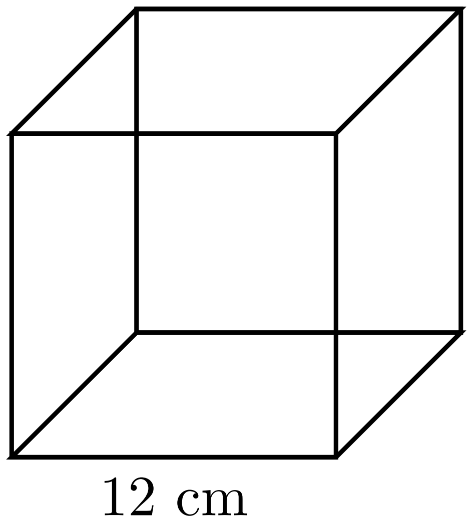
There are 6 faces therefore the surface area of the cube is;

$$= 6 \times 64\,\text{cm}^2$$
$$= 384\,\text{cm}^2$$

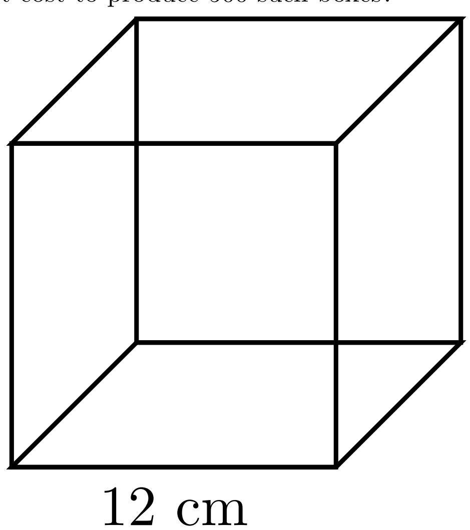
The surface area of the cube is 384 cm^2 □

Exercise

1.A large wooden dice with sides measuring 12 cm each is to be painted on all six faces. If one milliliter of paint covers 5 cm^2, calculate the total amount of paint required to cover the dice completely.



2. A cube has a side length of 25 cm. If the cost of the cardboard material is Ksh.110 per square centimeter, determine the total cost of making one box. How much would it cost to produce 500 such boxes?



3. A decorative gift box is shaped like a cube with each side measuring 18 cm. To wrap it completely, calculate the total area of wrapping paper needed.

If the shop sells wrapping paper in sheets of 1 square meter, how many sheets will be required to wrap 20 such gift boxes?

4. A small storage room is designed in the shape of a cube with an edge length of 4 meters. The floor, four walls, and ceiling all need to be tiled. If each tile covers an area of 0.5 square meters, determine the total number of tiles required to fully cover the interior of the room

5. A cube-shaped metal water tank with a side length of 2.5 meters is being coated with a protective layer on all its surfaces to prevent rusting. If the coating material costs Ksh.150 per square meter, calculate the total cost to coat the entire tank.

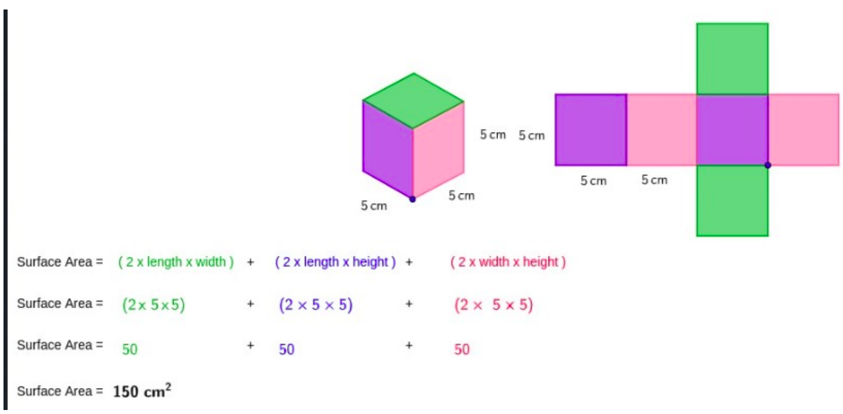
6. A pet shop wants to construct a cube-shaped aquarium with a side length of 1.2 meters. The aquarium needs to be made entirely of glass, including the base and all four vertical sides, **but the top will remain open**. If the cost of glass is Ksh. 750 per square meter, find the total cost of constructing the aquarium.

7. A company is designing a cube-shaped promotional stand with a side length of 5 meters. The stand will be covered with high-quality wallpaper on all six faces. If one roll of wallpaper covers 10 square meters, determine the number of rolls required to fully cover the cube.

2.7.2 Surface area of a cuboid

Activity 2.7.3 Surface area of a Cuboid

View the net of a cuboid in order to understand how to find the surface area of the cuboid.



Activity 2.7.4 Brick wall construction..

Materials needed;

Small bricks

Rulers

Worksheets

In your group build up a cuboid wall by stacking similar small bricks to a desired height.

Add more columns and rows until the wall is entirely covered with bricks and it resembles a box.

Count the number of bricks that you used and record.

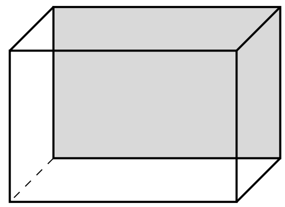
Calculate the surface area of the cuboid using the formula: $2(\mathrm{l} \times \mathrm{w}) + 2(\mathrm{l} \times \mathrm{h}) + 2(\mathrm{w} \times \mathrm{h})$

Compare their results. What do you notice? Share with your group members and discuss why builders need surface area e.g. for painting or tiling, building up walls and houses etc.

Key Takeaway

A cuboid is also called a rectangular Prism.

Rectangular Prism; It has a rectangle at its base. A cube is a rectangular prism with all sides of equal length.

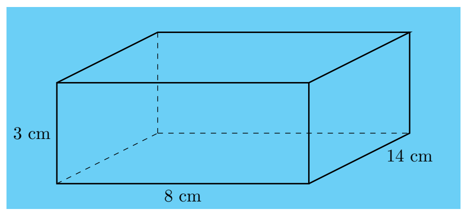


The surface area of a cuboid is the total area of all six faces of the cuboid.

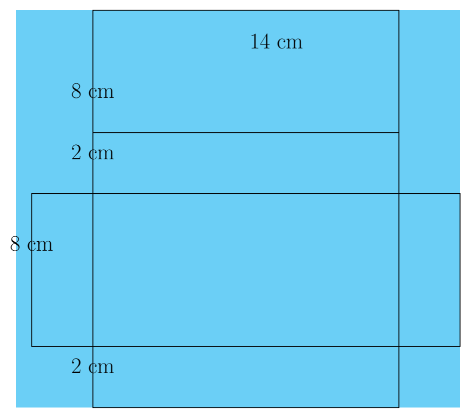
When learning about area, we calculated the surface area of a cuboid. Since the opposite faces of a cuboid are identical, the **surface area of a cuboid** can be calculated by finding the **area of each face** and **then adding them together**.

In this section, we will calculate the surface area of a cuboid from their nets.

Example 2.7.2 Find the surface area of the following rectangular prism:



Solution. Sketch and label the net of the prism.



Find the areas of the different shapes in the ne

$$\begin{aligned}\text{large rectangle} &= \text{perimeter of small rectangle} \times \text{length}\\ &=(3+8+3+8)\text{cm} \times 14\text{cm}\\ &=22\text{cm} \times 14\text{cm}\\ &=308\text{cm}^2\end{aligned}$$

$$\begin{aligned}2\,\text{small rectangle} &= 2(8\text{cm} \times 3\text{cm})\\ &=2(18)\,\text{cm}\\ &=36\text{cm}^2\end{aligned}$$

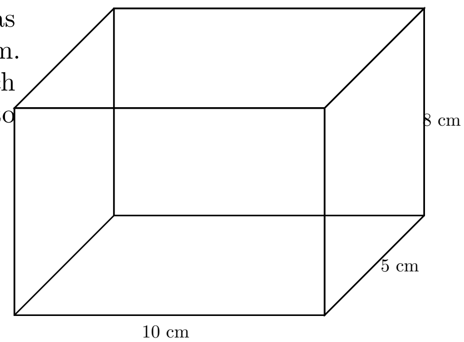
Find the sum of the areas of the faces

$$\begin{aligned}\text{large rectangle} + \text{small rectangle} =& (308 + 36)\text{cm}^2\\ =& 344\,\text{cm}^2\end{aligned}$$

The surface area of the rectangular prism is $344\,\text{cm}^2$ □

Exercise

1. A rectangular cardboard box has dimensions of 10 cm by 8 cm by 5 cm. Calculate its total surface area, which represents the total material required to construct the box.

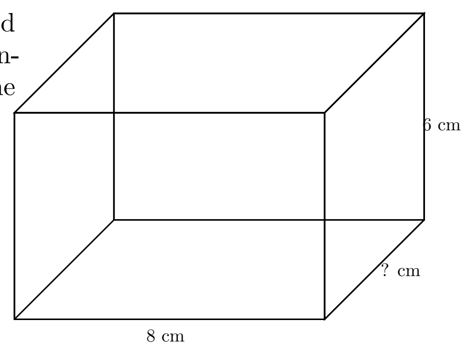


2. (a) A gift shop sells a rectangular gift box with dimensions 30 cm by 20 cm by 12 cm. If the shop owner wants to wrap the entire box, including all its faces, calculate the minimum amount of wrapping paper needed. If the wrapping paper is sold in rolls of 1 square meter, how many rolls would be needed to wrap 50 boxes?

(b) A gift box is being wrapped for a special occasion, and it has dimensions of 20 cm in length, 12 cm in width, and 10 cm in height. Calculate the exact amount of wrapping paper required to cover the entire box without any overlap.

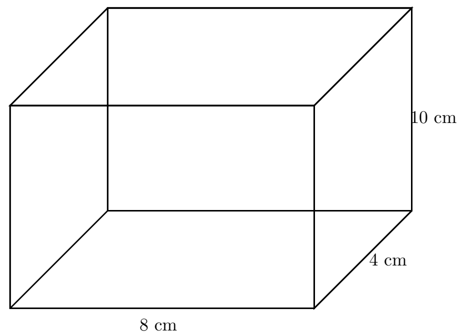
3. A metal box used for shipping measures 25 cm by 15 cm by 10 cm. Compute the total amount of sheet metal required to construct the box, assuming no material is wasted.

4. The total surface area of a cuboid is given as $484\ \text{cm}^2$ and two of its dimensions are 8 cm and 6 cm. Determine the missing height (h) of the cuboid.



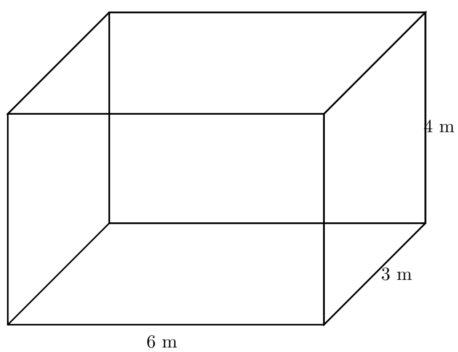
5. A cuboidal storage room has dimensions of 4 m by 5 m by 3 m. If the walls, floor and ceiling need to be painted, determine the total area that will be covered with paint.

6. A rectangular classroom has a length of 10 meters, a width of 8 meters and a height of 4 meters. The four walls and the ceiling need to be painted, but the floor is covered with tiles. If one litre of paint covers 5 square meters, calculate the total area to be painted and determine the amount of paint required.



7. A company is designing a cuboidal packaging box with dimensions 25 cm by 15 cm by 10 cm. The company wants to reduce costs by using the minimum possible material while ensuring the entire box is covered. Calculate the total surface area of the box and determine the cost of producing 1,000 such boxes if the material costs Ksh. 135 per square centimeter.

8. A swimming pool in the shape of a cuboid has dimensions 12 meters in length, 5 meters in width, and 3 meters in depth. The interior of the pool, including the bottom and the four walls, needs to be covered with waterproof tiles. If each tile has an area of 0.25 square meters, determine the total number of tiles required to completely cover the pool's surface.



9. A metal storage container is shaped like a cuboid with dimensions 6 m by 4 m by 3 m. The container needs to be insulated on all its surfaces except for one of the 6 m by 4 m walls, which serves as the entrance. If the insulation material costs Ksh.295 per square meter, determine the total cost of insulating the container.

Checkpoint 2.7.3 The figure below represents a building block with dimensions.var board = JXG.JSXGraph.initBoard('box', { boundingbox: [-5, 15, 20, -5], axis: false, showNavigation: false, showCopyright: false }); // Front face (12 cm x 4 cm) var A = board.create('point', [2, 2], { visible: false ,fixed:true }); var B = board.create('point', [14, 2], { visible: false ,fixed:true }); var C = board.create('point', [14, 6], { visible: false ,fixed:true }); var D = board.create('point', [2, 6], { visible: false ,fixed:true }); // Back face (shifted for 3D effect) var E = board.create('point', [5, 5], { visible: false ,fixed:true }); var F = board.create('point', [17, 5], { visible: false ,fixed:true }); var G = board.create('point', [17, 9], { visible: false ,fixed:true }); var H = board.create('point', [5, 9], { visible: false ,fixed:true }); // Draw front and back faces var frontFace = board.create('polygon', [A, B, C, D], { borders: { strokeColor: 'blue' ,fixed:true } }); var backFace = board.create('polygon', [E, F, G, H], { borders: { strokeColor: 'blue' ,fixed:true } }); // Connect front and back faces with solid lines board.create('line', [A, E], { straightFirst: false, straightLast: false, strokeColor: 'blue' ,fixed:true }); board.create('line', [B, F], { straightFirst: false, straightLast: false, strokeColor: 'blue' ,fixed:true }); board.create('line', [C, G], { straightFirst: false, straightLast: false, strokeColor: 'blue' ,fixed:true }); board.create('line', [D, H], { straightFirst: false, straightLast: false, strokeColor: 'blue' ,fixed:true }); // Labels for dimensions board.create('text', [(A.X() + B.X()) / 2, A.Y() - 0.8, "33 cm"],{fixed:true });

board.create('text', [B.X() +2.3, (B.Y() + C.Y()) / 2, "39 cm"],{fixed:true });
board.create('text', [(G.X() + F.X()) / 2 +0.5, (G.Y() + F.Y()) / 2 + 0.9, "26 cm"],{fixed:true });

1. Calculate the volume of the block.

   Volume =_________ cm^3

2. Calculate the total surface area of the block.

   Surface area =_________ cm^2

3. If 10 such blocks are packed together to form a large cube-shaped structure, what is the total volume of the structure?

   Total volume =_________ cm^2

Answer 1. 33462

Answer 2. 6318

Answer 3. 334620

Solution. Given:
A building block has:

- Length = 33 cm
- Width = 39 cm
- Height = 26 cm

There are 10 such identical blocks.

1. Find the volume of one block

   We use the formula for the volume of a cuboid:

   Volume = Length × Width × Height

   Where:

   $l = 33\,\text{cm}$

   $w = 39\,\text{cm}$

   $h = 26\,\text{cm}$

   Substitute the given values:

$$\begin{aligned}\text{Volume} &= 33 \times 39 \times 26 \\ &= 33462\,\text{cm}^3\end{aligned}$$

   Therefore,

   The volume of the building block = 33462 cm^3

2. Find the total surface area of one block

   The surface area of a cuboid is calculated using:

   Surface Area = $2(lw + wh + hl)$

$$\begin{aligned}\text{Surface Area} &= 2\left((33 \times 39) + (39 \times 26) + (33 \times 33)\right) \\ &= 2(1287 + 1014 + 858)\end{aligned}$$

$$= 2 \times 3159$$
$$= 6318\,\text{cm}^2$$

Therefore,

The surface area of the building block is $6318\,\text{cm}^2$

3. Find the total volume of all 10 blocks

   We already found the volume of one block is: $= 33462\,\text{cm}^3$

   Now multiply by the number of blocks:

$$\begin{aligned}\text{Total Volume} &= 33462 \times 10\\ &= 334620\,\text{cm}^3\end{aligned}$$

Therefore,

Total volume of 10 blocks $= 334620\,\text{cm}^3$

Checkpoint 2.7.4 This question contains interactive elements.

Checkpoint 2.7.5 A rectangular chalk box with no lid has dimensions of 12 cm in length, 47 cm in width, and 44 cm in height. What is its total external surface area?

Surface area $=$_________cm^2

Answer. 5756

Solution. For a rectangular open box (open at the top), the surface area consists of:

1. Base (bottom): $L \times W$
2. Two sides: $2(L \times H)$
3. Front and back: $2(W \times H)$

Therefore the formula is:

Surface Area $= (L \times W) + 2(L \times H) + 2(W \times H)$

Given:

1. length $= 12\,\text{cm}$
2. width $= 47\,\text{cm}$
3. height $= 44\,\text{cm}$

Substitute values in the formula.

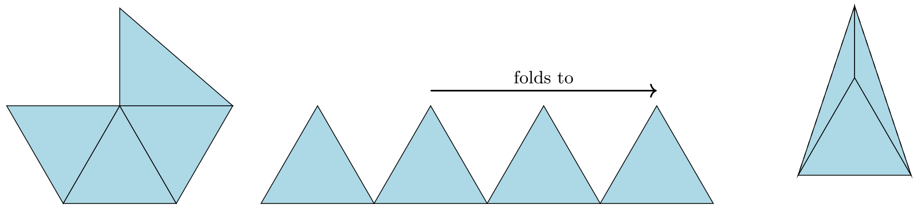
$$\begin{aligned}\text{Surface Area} &= (12 \times 47) + 2(12 \times 44) + 2(47 \times 44)\\ &= 564 + 2(528) + 2(2068)\\ &= 564 + 1056 + 4136\\ &= 5756\,\text{cm}^2\end{aligned}$$

Therefore,

The external surface area of the open chalk box is $5756\,\text{cm}^2$

2.7.3 Surface Area of a pyramid

Activity 2.7.5 Net of a triangle based Pyramid.
Paper Folding to form a Pyramid.



Triangular Pyramid

Net 1 Net 2

Materials Needed.

- Colored paper or cardstocks
- Scissors
- Rulers
- Glue or tape
- Markers
- Steps.

  Observe the net of the pyramid shown above.

  Draw on the Colored paper or cardstocks the net of a pyramid that is, four triangles. Note the sides of the triangles and height should be equal.

  Cut off the extra paper to remain with the pyramid net.

  Fold along the edges to form a pyramid. Hold the triangles in position using a tape.

  Using a ruler, measure the base height and slant height of the triangular faces.

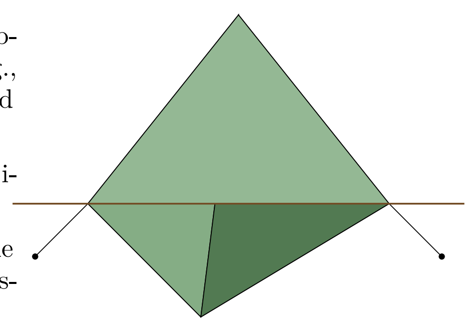
- Find the volume of the pyramid you have constructed.

Activity 2.7.6 Observing pyramidal objects in the surrounding environment.

- Identify pyramid-shaped objects around school/home e.g., tent(Shown alongside), roof, food container, toys etc.

  Measure dimensions or use estimated values.

  Find their surface area using the formula and compare with classmates.



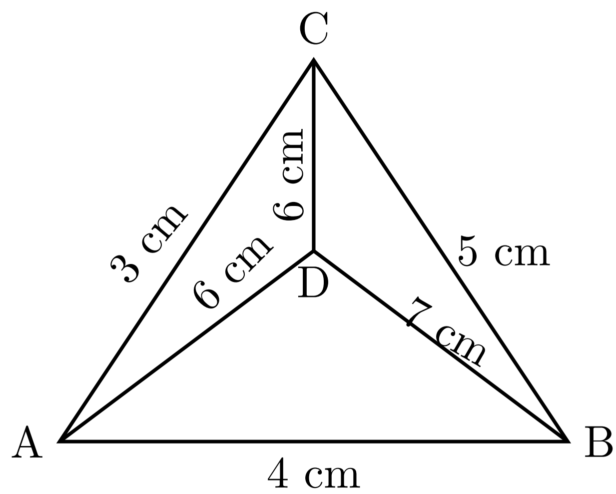
A pyramid is a geometric solid object that has a polygon as its base and faces that converge at a point called the apex. In other words the faces are not perpendicular to the base

The polygons that can act as the face of the pyramids include:
Triangle: Thus called triangular pyramid.
Squares: Thus called square pyramid.

The two pyramids take their names after the shape of their base.

A right pyramid's line between the apex and the centre of the base is perpendicular to the base.

Example 2.7.6 Find the surface area of the following triangular pyramid (correct to 1 d.p)



Solution. The surface area of a triangular pyramid is the sum of the areas of all **four triangular faces** □

In finding the surface area we need to recall the definations of a net and a prism.

The net of a pyramid consists of a base triangle ABC, which is a polygon that represents the base of the pyramid, and triangular faces$\angle ABD$, $\angle BCD$, $\angle ACD$ that represent the sides of the pyramid.

Each triangle's area will be found using Heron's Formula:

$$A = \sqrt{S(S-a)(S-b)(S-c)}$$

Where S is the semi perimeter.

$$S = \frac{a+b+c}{2}$$

Find the Area of the Base Triangle ABC. Given sides $AB = 4\,\text{cm}$, $BC = 5\,\text{cm}$, $AC = 3\,\text{cm}$

Calculate the semi-perimeter:

$$S = \frac{4\,\text{cm} + 5\,\text{cm} + 3\,\text{cm}}{2} = 6\,\text{cm}$$

Apply Heron's Formula:

$$\begin{aligned} A_{ABC} &= \sqrt{6(6-4)(6-5)(6-3)} \\ &= \sqrt{6 \times (2) \times (1) \times (3)} \\ &= \sqrt{36} \\ &= 6\,\text{cm}^2 \end{aligned}$$

Find the Area of Side TriangleABD Given sides $AB = 4\,\text{cm}$, $AD = 6\,\text{cm}$, $BD = 7\,\text{cm}$

$$S = \frac{4\,\text{cm} + 6\,\text{cm} + 7\,\text{cm}}{2} = 8.5\,\text{cm}$$

$$\begin{aligned} A_{ABD} =& \sqrt{8.5(8.5-4)(8.5-6)(8.5-7)} \\ =& \sqrt{8.5 \times (4.5) \times (2.5) \times (1.5)} \\ =& \sqrt{143.44} \\ =& 11.97\,\text{cm}^2 \end{aligned}$$

Find the Area of Side TriangleBCD Given sides $BC = 5$, cm $CD = 6$, cm $BD =$ 7cm

$$\begin{aligned} S =& \frac{5\,\text{cm} + 6\,\text{cm} + 7\,\text{cm}}{2} = 9\,\text{cm} \\ A_{BCD} =& \sqrt{9(9-4)(9-6)(9-7)} \\ =& \sqrt{9 \times (4) \times (2) \times (3)} \\ =& \sqrt{216} \\ =& 14.7\,\text{cm}^2 \end{aligned}$$

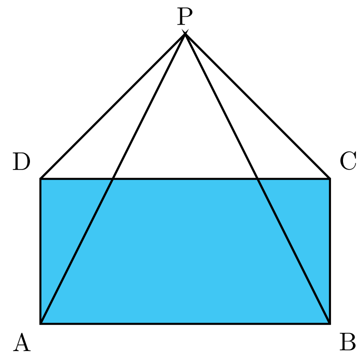
Find the Area of Side TriangleACD Given sides $AC = 3$, $AD = 6$, $CD = 6$

$$\begin{aligned} S =& \frac{3\,\text{cm} + 6\,\text{cm} + 6\,\text{cm}}{2} = 7.5\,\text{cm} \\ A_{ACD} =& \sqrt{7.5(7.5-4)(7.5-6)(7.5-7)} \\ =& \sqrt{7.5 \times (4.5) \times (1.5) \times (1.5)} \\ =& \sqrt{75.94} \\ =& 8.71\,\text{cm}^2 \end{aligned}$$

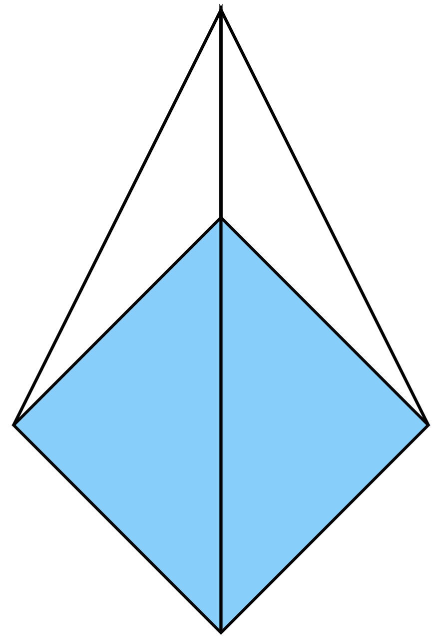
Find the Total Surface Area. Surface Area $= A_{ABC} + A_{ABD} + A_{BCD} + A_{ACD}$

$$\begin{aligned} =& 6\,\text{cm}^2 + 11.97\,\text{cm}^2 + 14.7\,\text{cm}^2 + 8.71\,\text{cm}^2 \\ =& 41.38\,\text{cm}^2 \end{aligned}$$

Example 2.7.7 Draw the net of the pyramid.

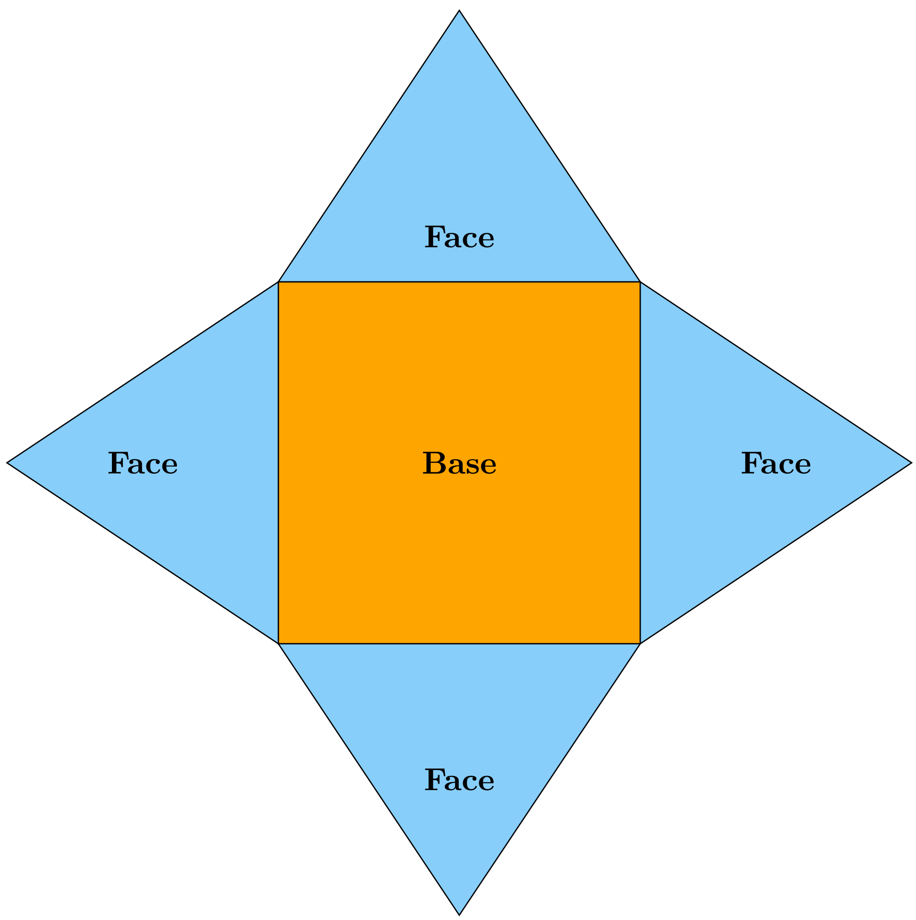


Rectangular Pyramid



Square Pyramid

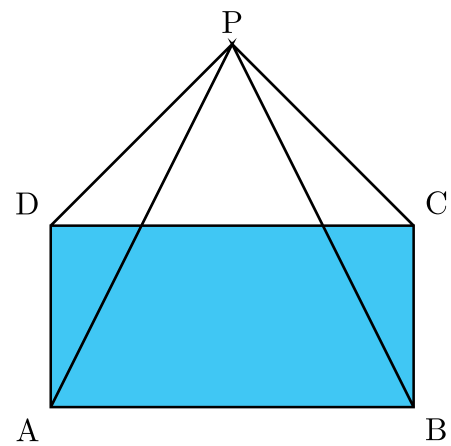
Solution. The net of the pyramid is shown below.



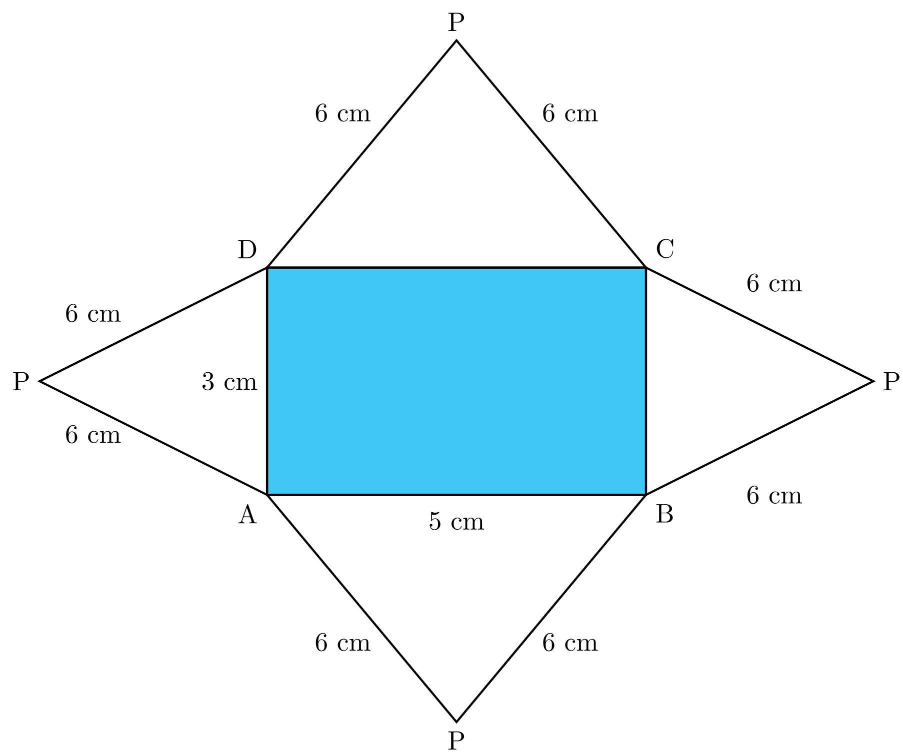
□

The surface area of a pyramid is the total area of the lateral faces and the base of the pyramid.

Example 2.7.8 Calculate the surface area of a pyramid whose base is rectangular and slant height 6cm as shown below.



Solution. First we form the net of the pyramid.



The area of the rectangular base

$$=\text{length} \times \text{width}$$
$$=5\,\text{cm} \times 3\,\text{cm}$$
$$=15\,\text{cm}^2$$

The height of triangles with base of 5 cm

$$=\sqrt{36 - 6.25}$$

$$
\begin{aligned}
&= \sqrt{29.75} \\
&= 5.45\,\text{cm}\,(2\text{d.p})
\end{aligned}
$$

The area of a triangle with base of 5 cm

$$
\begin{aligned}
&= \frac{1}{2} \times \text{base} \times \text{height} \\
&= \frac{1}{2} \times 5 \times 5.45 \\
&= 13.625\,\text{cm}^2
\end{aligned}
$$

Since this triangles are two,Therefore total area is:

$$
\begin{aligned}
&= 2 \times 13.625 \\
&= 27.25\,\text{cm}^2
\end{aligned}
$$

The height of triangles with base of 3 cm

$$
\begin{aligned}
&= \sqrt{6^2 - 1.5^2} \\
&= \sqrt{36 - 2.25} \\
&= \sqrt{33.75} \\
&= 5.81\,\text{cm}^2(2\text{d.p})
\end{aligned}
$$

The area of a triangle with base of 3 cm

$$
\begin{aligned}
&= \frac{1}{2} \times \text{base} \times \text{height} \\
&= \frac{1}{2} \times 3 \times 5.81\,\text{cm}^2 \\
&= 8.715\,\text{cm}^2
\end{aligned}
$$

Since this triangles are two,Therefore total area

$$
\begin{aligned}
&= 2 \times 8.715 \\
&= 17.43\,\text{cm}^2
\end{aligned}
$$

Therefore, the surface area of the pyramid is,

$$
\begin{aligned}
&= 15\,\text{cm}^2 + 27.25\,\text{cm}^2 + 17.43\,\text{cm}^2 \\
&= 59.68\,\text{cm}^2
\end{aligned}
$$

□

Example 2.7.9 Find the surface area of a square pyramid with a height of 6 cm and a side length of 4cm.

Solution. Select the correct formula and substitute with the given values.

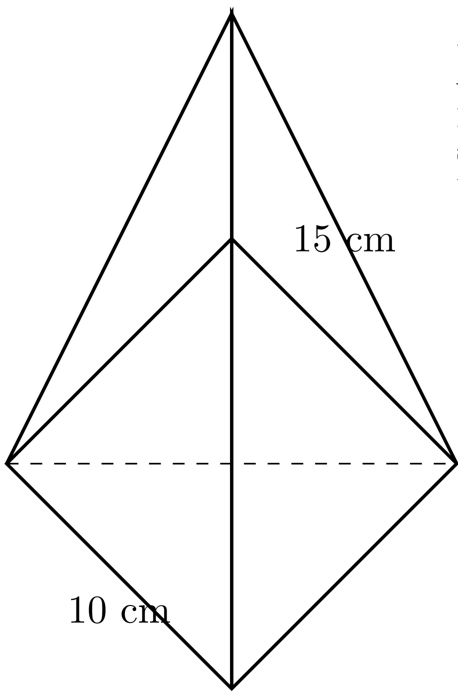
We are given b = 2 by 2 and H = 4, therefore

$$
\begin{aligned}
\text{Surface Area} &= \text{Base area} + 4(\text{area of triangles}) \\
\text{Base Area} = 2\,\text{cm} \times 2\,\text{cm} \\
&= 4\,\text{cm}^2 \\
\text{Area of one triangle} &= \frac{1}{2} b \times h
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2} \times 2\,\text{cm} \times 4\,\text{cm} \\
&= 4\,\text{cm}^2 \\
\text{Area of all triangles} &= 4\,\text{cm}^2 \times 4\,\text{triangles} \\
&= 16\,\text{cm}^2 \\
\text{Surface area} &= 16\,\text{cm}^2 + 4\,\text{cm}^2 \\
&= 20\,\text{cm}^2
\end{aligned}
$$

The surface area for the square pyramid is $20\,\text{cm}^2$. □

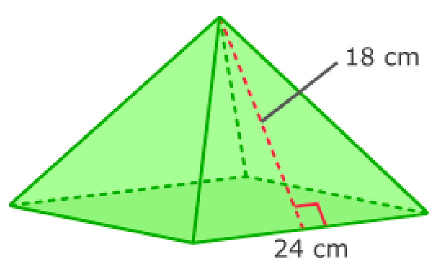
Exercise



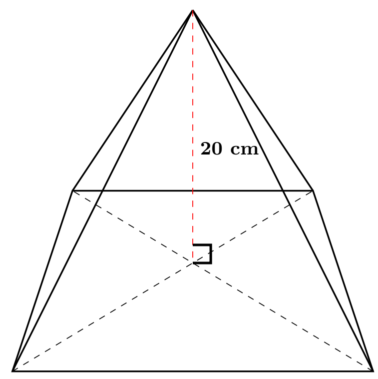
1. A square-based pyramid has a base with sides measuring 10 cm each, while its triangular faces have a slant height of 15 cm. Determine the total surface area of this pyramid, including the base and all four triangular faces.

2. A pyramid with a square base has a total surface area of 400 cm^2, and its base side measures 8 cm. Using the formula for surface area, calculate the slant height of the pyramid.

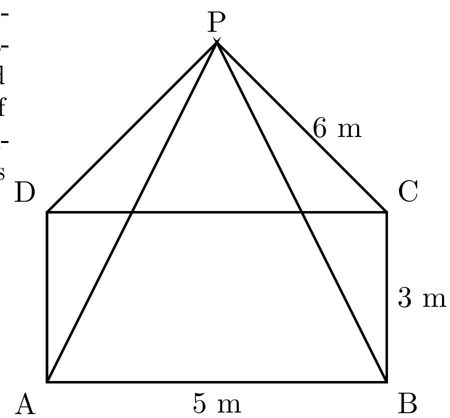
3. A square pyramid has a base with sides of 24 cm each, and the height of the triangular face is 18 cm. Find the total surface area of the pyramid.



4. A miniature paper pyramid is being designed with a square base of 12 cm by 12 cm and a slant height of 20 cm. How much paper is required to construct the entire pyramid?



5. The roof of a small storage building is in the shape of a rectangular pyramid with a base side length of 5 m and a slant height of 6 m. If the entire roof needs to be covered with wooden shingles, calculate the total area that needs to be covered.



Checkpoint 2.7.10 This question contains interactive elements.

Checkpoint 2.7.11 This question contains interactive elements.

2.7.4 Surface Area of a Sphere

Activity 2.7.7 Fun Activity Idea

Use an orange or a ball and cover it with small square sticky notes.

Estimate how many squares fit over the sphere's surface.

Then compare with other group members' results to the actual formula!

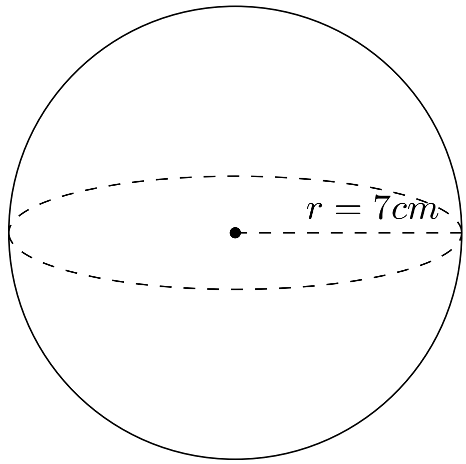
The surface area of a sphere is the total area covering its curved outer surface.

Formula for Surface Area of a Sphere

Surface Area $= 4\pi r^2$ where r is the radius of the sphere.

The $= 4\pi r^2$ comes from integrating small patches over the sphere's curved surface. We can compare the sphere to how a sphere fits inside a cylinder of the same radius and height.

Example 2.7.12 Find the surface area of the following sphere (correct to 1 decimal place



Solution. Surface area of a sphere $= 4\pi r^2$

$$
\begin{aligned}
&= \frac{22}{7} \times 4 \times (7\,\text{cm})^2 \\
&= \frac{22}{7} \times 196\,\text{cm}^2 \\
&= 22 \times 28\,\text{cm}^2 \\
&= 616\,\text{cm}^2
\end{aligned}
$$

□

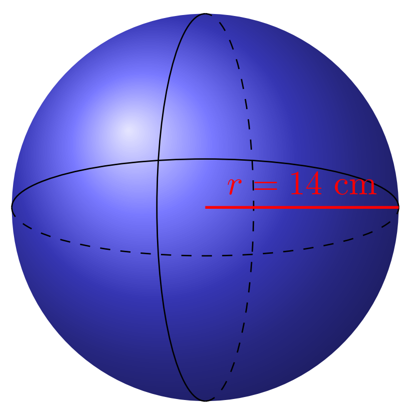
Example 2.7.13

1. A sphere has a radius of 14 cm.

   a) Find its total surface area.

   b) If the sphere were covered with paint, how much area would be painted?

   c)If a second sphere has twice the radius, how does its surface area compare to the first sphere?



Sphere with radius $r = 14cm$

Hint. Remember, if the radius doubles, the surface area increases by 4 times (since $(2r)^2 = 4\pi r^2$.

Solution. a) Total Surface area.

$$
\begin{aligned}
&= 4 \times \frac{22}{\not 7} \times \not{14}\,\text{cm} \times 14\,\text{cm} \\
&= 4 \times 22 \times 2\,\text{cm} \times 14\,\text{cm} \\
&= 4 \times 22 \times 28 \\
&= 4 \times 616\,\text{cm}^2 \\
&= 2464\,\text{cm}^2
\end{aligned}
$$

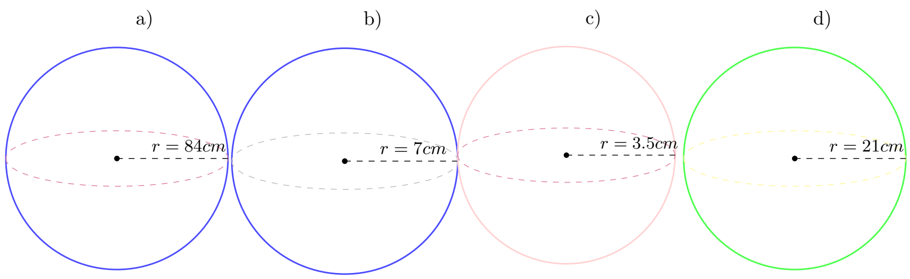
The surface area of the sphere is $2464\,\text{cm}^2$.
b) The **area of the sphere painted** would be the entire surface area.
so the Painted area $=2464\,\text{cm}^2$.
c) If the radius is doubled i.e,
($r = 14\,\text{cm} + 14\,\text{cm} = 28\,\text{cm}$),
the new surface area would be:

$$\begin{aligned}\text{New surface area} =& 4\frac{22}{\cancel{7}} \times \cancel{28}\,\text{cm} \times 28\text{cm}\\ =& 4 \times 22 \times 4\,\text{cm} \times 28\,\text{cm}\\ =& 4 \times 22 \times 112\text{cm}^2\\ =& 4 \times 616\,\text{cm}^2\\ =& 9856\,\text{cm}^2\end{aligned}$$

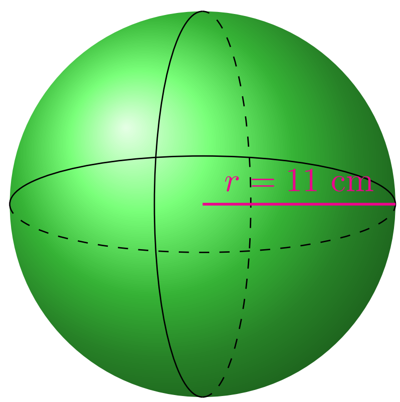
The surface area of the sphere is $9856\,\text{cm}^2$ if the radius is doubled. □

Exercise

Find the surface areas of the figure below.



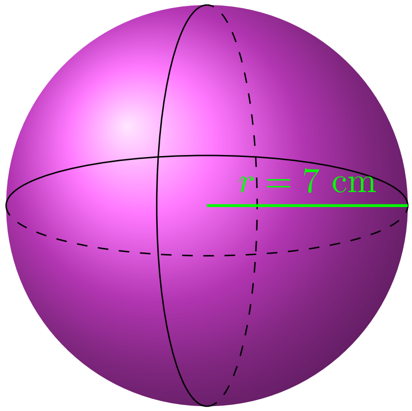
1. A football used in a tournament has a radius of 11 cm. Find the total surface area of the football, assuming it is a perfect sphere.



Sphere with radius $r = 11cm$

2. The surface area of a spherical ornament is measured to be 452.16 cm². Using the formula for the surface area of a sphere, determine the radius of the ornament.

3. A planetarium is constructing a dome in the shape of a hemisphere with a radius of 20 m. Since the dome covers only half of a full sphere, determine its total surface area, including the flat circular base.



Sphere with radius $r = 7cm$

4. A spherical metal ball with a radius of 7 cm is to be coated with a layer of paint. Determine the total area that needs to be covered with paint.

5. A company is designing a spherical water tank with a diameter of 24 cm. Compute the total surface area of the tank, which represents the external surface that will be painted.

Checkpoint 2.7.14 This question contains interactive elements.

Checkpoint 2.7.15 A company is designing a spherical water tank. The total external surface area of the tank, which will be painted, is $199584\,\text{m}^2$. Using this information, determine the diameter of the spherical tank.

Diameter of the water tank =_______ m

Answer. 252

Solution. We are asked to determine the ***diameter*** of a spherical water tank given its total surface area.

Step 1: Recall the formula for the surface area of a sphere

Surface area $= 4\pi r^2$

Step 2: Substitute the given surface area

The total surface area is $199584\,\text{m}^2$ and $\pi = \dfrac{22}{7}$. Let r be the radius of the tank.

$$\begin{aligned} 4\pi r^2 &= 199584 \\ 4 \times \frac{22}{7} \times r^2 &= 199584 \\ \frac{88}{7} r^2 &= 199584 \\ r^2 &= \frac{7 \times 199584}{88} \\ r &= \frac{\sqrt{7 \times 199584}}{\sqrt{88}} \\ &= 126\,\text{m} \end{aligned}$$

Step 3: Compute the diameter

$$\begin{aligned} \text{Diameter} &= 2r \\ &= 2 \times 126 \\ &= 252\,\text{m} \end{aligned}$$

2.7.5 Surface Area of a Triangular Prism

Activity 2.7.8 Formula of the surface area of a triangular prism is the sum of:

1. Two triangular bases

$$\text{Area of a triangle} = \frac{1}{2} \times \text{base} \times \text{height}$$

$$\begin{aligned}\text{Total area for two triangles}\\ &= 2 \times (\frac{1}{2} \times \text{base} \times \text{height})\\ &= b \times h\end{aligned}$$

2.Two rectangular lateral faces

The three faces depend on the perimeter of the triangular base and the prism length L

$$\begin{aligned}\text{Lateral Area} &= \text{Base Area} \times \text{Lateral Area}\\ &= (b \times h) + (\text{Perimeter} \times L)\end{aligned}$$

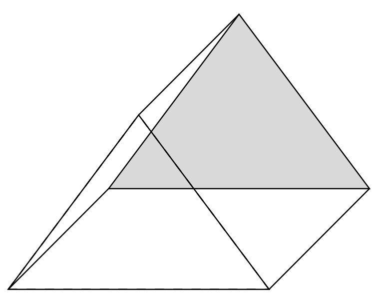
Total Surface Area formula:

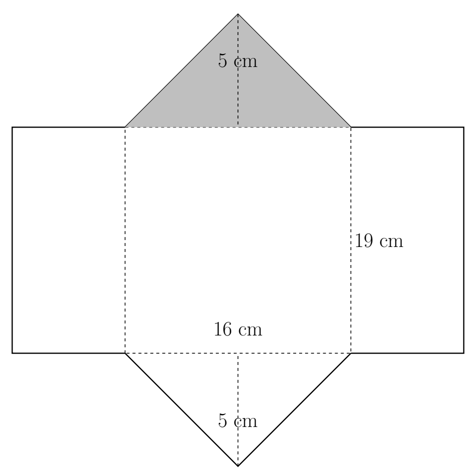
$$S.A = \text{Base Area} \times \text{Lateral Area}$$

Key Takeaway

Triangular Prism; It has a triangle at its base.

A triangular prism is a geometric object with two identical triangular bases and three rectangular lateral faces. Its surface area is the total area of all its faces, measured in square units (cm^2, m^2, etc.).





Example 2.7.16 Find the Surface area of the triangular prism with a slant height of 5 cm, height of the triangular prism 12cm and a base of 8 cm.

Solution. Step 1: Find the area of the base.

The triangle has a slant height of 5 cm and base 8 cm. Using pythagorean relationship height is:

$$\begin{aligned}&= \sqrt{(5\,\text{cm})^2 - (4\,\text{cm})^2}\\ &= \sqrt{25\,\text{cm} - 16\,\text{cm}} = 3\,\text{cm}\end{aligned}$$

$$\text{area of a triangle} = \frac{1}{2} \times b \times h$$

$$=(\frac{1}{2} \times 3\,\text{cm} \times 8\,\text{cm}) \times 12\,\text{cm}$$
$$=144\,\text{cm}^2$$

Step 2: Multiply the area of the base by the height of the solid to find the volume

$$\begin{aligned}\text{volume} =& \text{area of base} \times \text{height}\\ =& \frac{1}{2} b \times h \times H\\ =& 48\,\text{cm}^2 \times 12\,\text{cm}\\ & 576\,\text{cm}^3\end{aligned}$$

The surface of the triangular prism is $576\,\text{cm}^3$. □

Exercise

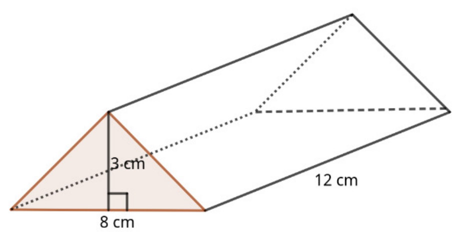
1.If a litre of paint covers an area of 2m^2, how much paint does a painter need to cover:

a) A rectangular swimming pool with dimensions 4 m by 3 m by 5 m (the inside walls and floor only);

b) the inside walls and floor of a circular reservoir with diameter 7 m and height 5 m

2. A triangular prism has a triangular base of 13 cm and the height of the prism 9 cm. Calculate the total surface area .

3. A prism is constructed with a triangular base of 8 cm height of 12 cm.. Determine the area of the triangular base, then use the given dimensions to compute the total surface area.



4. The total surface area of a triangular prism is measured as 360 cm^2 and its height is 20 cm. If the triangular base has sides measuring 9 cm, 12 cm and 15 cm, verify that this value is correct by calculating the surface area from scratch.

5. A bridge support structure has the shape of a triangular prism, with a base measuring 10 cm, 17 cm and 21 cm, and a height of 50 cm. Compute the total surface area, which will help determine how much paint is needed to coat its entire surface.

6. A glass showcase is designed in the shape of a triangular prism, with a triangular base of 5 cm, a height of 12 cm, and a prism length of 20 cm. If all faces are to be made of glass, calculate the total glass area required.

Checkpoint 2.7.17 This question contains interactive elements.

Checkpoint 2.7.18 Look at the cube shown below. Each edge of the cube measures 7 cm.const board = JXG.JSXGraph.initBoard('box', { boundingbox: [-2, 5, 5, -2], showCopyright: false, axis: false, showNavigation: false, keepAspectRatio:true }); const side = 2.5; // Define points let A = board.create('point', [0, 0], {visible: false}); let B = board.create('point', [side, 0], {visible: false}); let C = board.create('point', [side, side], {visible:

false}); let D = board.create('point', [0, side], {visible: false}); let A1 = board.create('point', [0.8, 1], {visible: false}); let B1 = board.create('point', [side + 0.8, 1], {visible: false}); let C1 = board.create('point', [side + 0.8, side + 1], {visible: false}); let D1 = board.create('point', [0.8, side + 1], {visible: false}); // Create edges board.create('polygon', [A, B, C, D], {borders: {strokeColor: 'blue'}}); board.create('polygon', [A1, B1, C1, D1], {borders: {strokeColor: 'blue'}}); board.create('line', [A, A1], {straightFirst: false, straightLast: false, strokeColor: 'blue'}); board.create('line', [B, B1], {straightFirst: false, straightLast: false, strokeColor: 'blue'}); board.create('line', [C, C1], {straightFirst: false, straightLast: false, strokeColor: 'blue'}); board.create('line', [D, D1], {straightFirst: false, straightLast: false, strokeColor: 'blue'}); // Label the 3 visible edges with 13 cm board.create('text', [1.2, -0.2, "7 cm"]); board.create('text', [3.5, 2, "7 cm"]); board.create('text', [3, 0.4, "7 cm"]); Calculate the total surface area of the cube in:

1. Square centimeters =_________cm^2
2. Square meters =_________m^2

Answer 1. 294

Answer 2. 0.0294

Solution. Given the side = 7 cm:

Surface Area (SA) of a cube = 6 × $(\text{side})^2$ or $5l^2$ where l is the side of the cube.

Substitute the value in the formula of the cube.

$$\begin{aligned} \text{SA} &= 6 \times 7^2 \\ &= 6 \times (7 \times 7) \\ &= 6 \times 7 \\ &= 294\,\text{cm}^2 \end{aligned}$$

Convert cm^2 to m^2
Since:
$1\,\text{m}^2 = 10\,000\,\text{cm}^2$
Therefore,

$$\begin{aligned} \text{Surface Area in}\,\text{m}^2 &= \frac{294}{10\,000} \\ &= \frac{147}{5000}\,\text{m}^2 \end{aligned}$$

Which can be simplified as, 0.0294 m^2
The surface of the cube in

1. Square centimeters =294 cm^2
2. Square meters $=\frac{147}{5000}$ m^2 or 0.0294 m^2

Checkpoint 2.7.19 This question contains interactive elements.

Checkpoint 2.7.20 This question contains interactive elements.

Checkpoint 2.7.21 A tent is shaped like a triangular prism. The triangular ends of the tent each have a base of 40 cm and a height of 21 cm. The length of the tent is 43 cm. Calculate the total surface area of the tent's canvas.

Total surface area of the tent's canvas =_____ cm^2

Answer. 3334

Solution. Given: A tent is shaped like a triangular prism with the following dimensions:

Base of right triangle = 20 m Height of triangle = 21 m Sloping side (hypotenuse) = 29 m Big base of tent = 40 m Length of tent = 43 m

1. ***Find the Total Surface Area of the Tent***

   Step 1. Find the area of one triangular end

$$\begin{aligned} A_{\triangle} &= \frac{1}{2} \times \text{base} \times \text{height} \\ &= \frac{1}{2} \times 40 \times 21 \\ &= 420\,\text{m}^2 \end{aligned}$$

   Step 2. Find the area of one rectangular side (the sloping panel)

$$\begin{aligned} A_{\text{rect}} &= \text{sloping side} \times \text{length} \\ &= 29 \times 43 \\ &= 1247\,\text{m}^2 \end{aligned}$$

   Step 3. Find the total surface area of the tent's canvas

$$\begin{aligned} \text{Total Surface Area} &= 2(A_{\triangle}) + 2(A_{\text{rect}}) \\ &= 2(420) + 2(1247) \\ &= 3334\,\text{m}^2 \end{aligned}$$

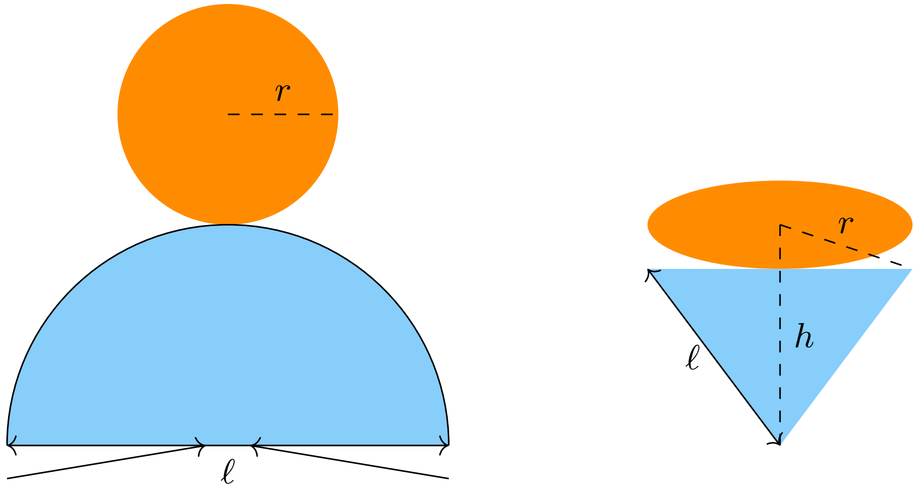
$$\boxed{\text{Total Surface Area of the Tent's Canvas} = 3334\,\text{m}^2}$$

2.7.6 Surface area of a Cone

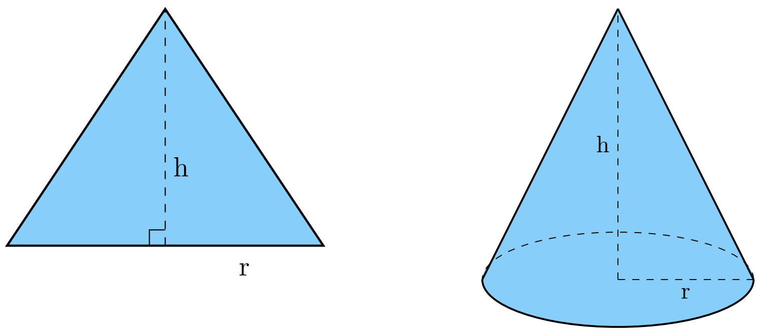
Activity 2.7.9

If a cone has a height of h and a base of radius r, show that the surface area is: $\pi r^2 + \pi r\sqrt{r^2 + h^2}$

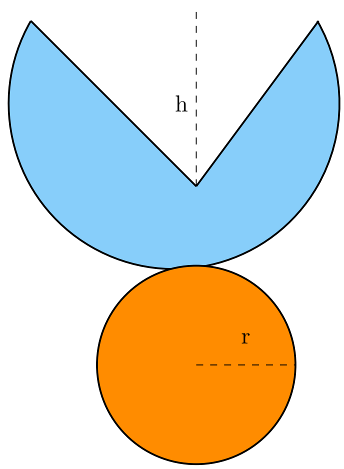
Sketch and label the cone



Identify the faces that make up the cone



The cone has two faces: the base and the faces makig up a wall. The base is a circle of radius r and the walls can be opened out to a semi-circle



This curved surface can be cut into many thin triangles with height close to a (where ℓ is the slant height). The area of these triangles or sectors can be summed as follow;

$$\begin{aligned} \text{Area} &= \frac{1}{2} \times \text{base} \times \text{height of a small triangle} \\ &= \frac{1}{2} \times 2\pi r \times \ell \end{aligned}$$

$$=\pi r\ell$$

ℓ can be calculated using the pythagorean theorem.
$\ell = \sqrt{r^2 + h^2}$
Calculate the area of the circular base $C_1 = \pi r^2$
Calculate the area of the curved walls $C_2 = \pi r\ell = \pi r\sqrt{r^2 + h^2}$
To find the surface area we sum up all the areas that is:

$$\begin{aligned} A =& C_1 + C_2 \\ =& \pi r^2 + \pi r\sqrt{r^2 + h^2} \\ =& \pi(r + \sqrt{r^2 + h^2}) \end{aligned}$$

The **net of a cone** is a two-dimensional representation of the three-dimensional shape of the cone. It is made up of the curved surface of the cone laid out flat, so that you can see the shape of the cone. The net of a cone is useful for visualizing the shape of the cone and for calculating its surface area and volume.

Example 2.7.22 Given a cone with the radius $r = 14\,\text{cm}$ and an angle of $\angle 60°$. Find the surface area of the cone.

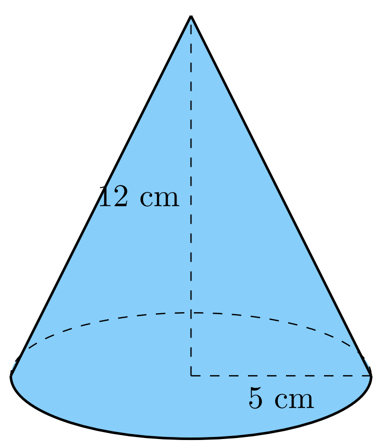
Solution.

$$\begin{aligned} \text{Area of sector A} =& \frac{\theta}{360°}\pi r^2 \\ =& \frac{60}{360°} \times \frac{22}{7} \times 14 \times 14 \\ =& 102.67\,\text{cm}^2 \\ \text{Area of circle B} =& \pi r^2 \\ =& \frac{22}{7} \times 14\,\text{cm} \times 14\,\text{cm} \\ =& 616\,\text{cm}^2 \\ \text{Surface area} =& 102.67\,\text{cm}^2 + 616\,\text{cm}^2 \\ =& 718.67\,\text{cm}^2 \end{aligned}$$

□

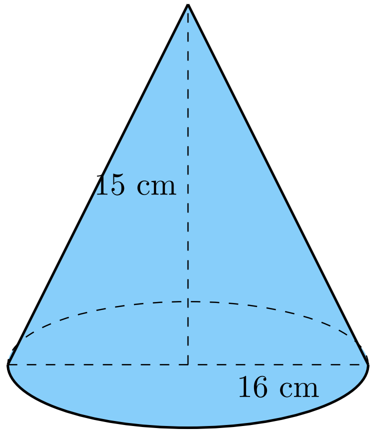
Exercise

1. A circular cone has a base radius of 5 cm and a slant height of 12 cm. Calculate the total surface area of the cone, including both the curved surface and the circular base



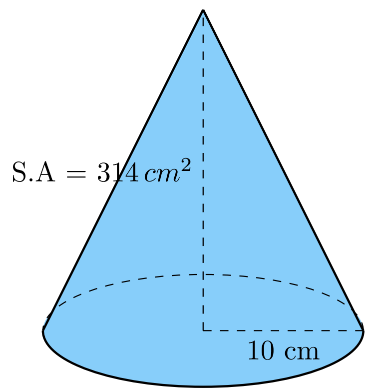
2. A cone is constructed with a base diameter of 16 cm and a height of

15 cm. Before finding the total surface area, determine the slant height of the cone using the Pythagorean theorem. Then, calculate the complete surface area.



3. A conical container, open at the top, is made of metal and has a base radius of 10 cm and a slant height of 18 cm. Determine the total metal sheet required to construct this container, excluding the base.

4. The total surface area of a cone is given as 314 cm², and its base radius is 10 cm. Using the surface area formula, determine the slant height of the cone.



5. A conical tent made of waterproof fabric has a radius of 4.2 m and a slant height of 7.5 m. If the tent does not have a base, calculate the area of fabric required to cover the tent completely.

Checkpoint 2.7.23 This question contains interactive elements.

Checkpoint 2.7.24 This question contains interactive elements.

2.7.7 Surface Area of Composite solids

Activity 2.7.10 Tree Model Surface Area

Theme: Modeling a tree using a cylinder $\backslash textbftrunk$ and hemisphere **tree top** or cone **pine tree top**

We will model a tree trunk as a composite solid, then calculate the total surface area, excluding the part where the top and trunk connect.

- **Materials needed:**
- Nets or templates for:

  Cylinder (tree trunk)

Cone or hemisphere (tree top)

- Scissors, glue or tape
- Rulers and string

Step-by-Step Instructions.

1. Create the Model

- Each student or group builds a tree model using
- A cylinder for the trunk
- Either a cone (pine tree) or a hemisphere (bushy tree) for the top

For example, a pine tree would be a cone on top of a cylinder.

2. Label Dimensions

- Measure and label your dimensions (or give them pre-set values). For example:

  Radius of trunk = 3 cm

  Height of trunk = 10 cm

  Radius of cone = 3 cm

  Slant height of cone = 5 cm

3. Surface Area Calculation

- Surface area of cylinder: Lateral area: $2\pi rh$ Bottom circle: $\pi r2$ Do NOT count the top circle — it is covered by the cone
- Surface area of cone:

  Lateral area: $2\pi r\ell$

  Do NOT count the base of the cone — it is attached to the trunk

  Add all visible surfaces:

  $\text{S.A}_{\text{total}}$ = **Lateral area of cone** + **Lateral area of cylinder** + **Base of cylinder**

- Sample Calculation:

  With the above values:

  Cylinder lateral: $2\pi \times (3\,\text{cm}) \times (10\,\text{cm}) = 60\,\text{cm} \times 3.14 = 188.40\,\text{cm}^2$

  Cylinder base: $\pi \times (3\,\text{cm})^2 = 9\,\text{cm} \times 3.14 = 28.26\,\text{cm}^2$

  Cone lateral: $\pi(3\,\text{cm}) \times (5\,\text{cm}) = 15\,\text{cm} \times 3.14 = 47.10\,\text{cm}^2$

  $S.A = 188.40\,\text{cm}^2 + 28.26\,\text{cm}^2 + 47.10\,\text{cm}^2 = 263.76\,\text{cm}^2$

- **Study Questions**

  Why don't we count the base of the cone or the top of the cylinder?

  What happens if the cone is bigger than the cylinder?

  How would the surface area change if the tree had branches modeled as small cylinders?

A **Solid** is a three dimensional shape. **Solids** are objects with three dimensions i.e Width, Length and Height and they have surface area and volumes..

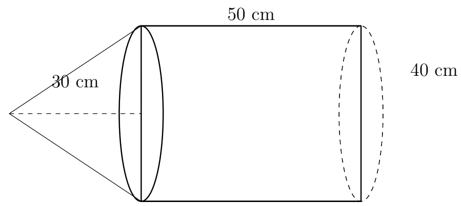
Area of composite solids

When two or more different solids are placed together, the result is composite solids. The surface area of a composite solids can be found by adding areas of the parts of the solids.

Example 2.7.25

Find the surface area of the figure alongside. Leave your answer in m^2

Solution.



Surface Area of a cone.

$$\begin{aligned} CSA_{\text{cone}} =& \pi r \times (\ell = \sqrt{r^2 + h^2}) \\ =& 3.14 \times (40\,\text{cm}) \times \sqrt{(40)^2\,\text{cm} + (30)^2\,\text{cm}} \\ =& 3.14 \times 40\,\text{cm} \times 50\,\text{cm} \\ =& 125.60 \times 50 \\ =& 6,280\text{cm}^2 \end{aligned}$$

Curved Surface Area of the Cylinder

$$\begin{aligned} CSA_{\text{cylinder}} =& 2\pi r \times h \\ =& 2 \times 3.14 \times (40\,\text{cm}) \times 50\ \ \text{cm} \\ =& 2 \times 3.14 \times 40\,\text{cm} \times 50\,\text{cm} \\ =& 251.20\,\text{cm}^2 \times 50\,\text{cm} \\ =& 12,560\,\text{cm}^2 \end{aligned}$$

Base Area of the Cylinder (since only the bottom is exposed)

$$\begin{aligned} \text{Base Area} =& \pi r^2 \\ =& \pi \times (40)^2\,\text{cm} \\ =& 5024\,\text{cm}^2 \end{aligned}$$

Total Surface Area $= CSA_{\text{cylinder}} + CSA_{\text{cone}} +$ Base Area

$$\begin{aligned} =& 12,560\,\text{cm}^2 + 6,280\,\text{cm}^2 + 5024\,\text{cm}^2 \\ =& 23,864\,\text{cm}^2 \end{aligned}$$

Converting to m^2

$$\begin{aligned} 1\text{m} =& 100\,\text{cm} \\ 1\text{m}^2 =& 10,000\,\text{cm}^2 \\ & ?23,864\,\text{cm}^2 \\ =& 1\,\text{m} \times \frac{23,864\,\text{cm}^2}{10,000\,\text{cm}^2} \\ =& 2.3864\,\text{m}^2 \\ =& 2.39\,\text{m}^2 \ \ \text{(to two d.p)} \end{aligned}$$

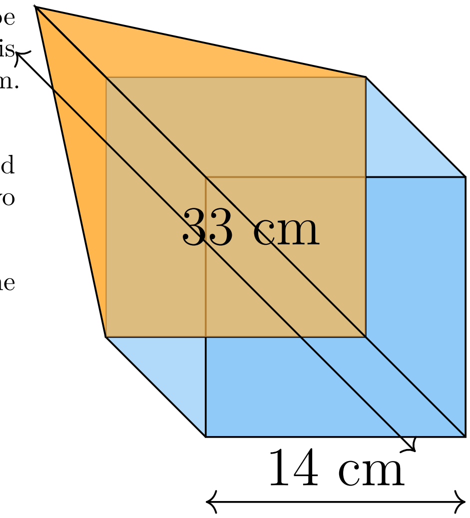
□

Exercise

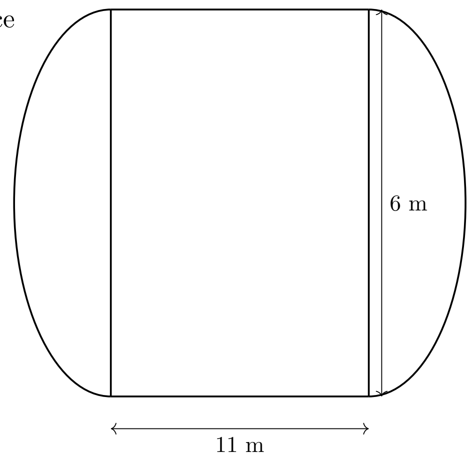
1. The solid below is made of a cube and a square pyramid whose height is 33cm and cube sides measures 14cm. Answer the following

a) Find the surface area of the solid shown. Give your answers to two decimal places.

b) Now determine the volume of the composite solids.



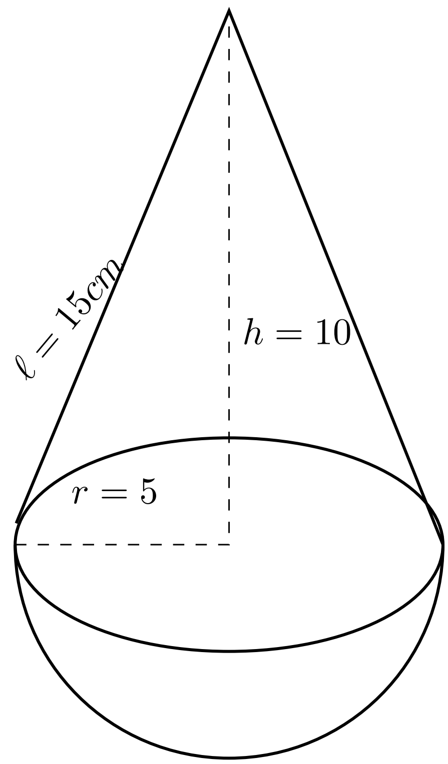
2. Calculate the volume and surface area of the solid alongside.



3. A right circular icecream cone with a radius of 3 cm and a height of 12 cm holds a half scoop of ice cream in the shape of a hemisphere on top. If the ice cream melts completely, will it fit inside the cone? Show all calculations to justify your answer.

4. A lampshade is in the shape of a frustum of a cone. The top and bottom circular openings have diameters of 12 cm and 20 cm, respectively. If the slant height is 15 cm, find the lateral surface area of the lampshade.

5. Mogaka a grade 10 student was trying to sketch an image of a ice cream cone container with icecream. Find the surface area of the sketched image alongside.



6.A birthday cake has a cylindrical base of radius 10 cm and height 15 cm. The top is shaped like a hemisphere with the same radius. Find the total volume of the cake.

7.A goblet consists of a hemisphere on top of a cylindrical base. The hemisphere has a radius of 5 cm, and the cylinder has the same radius with a height of 12 cm. Find the surface of the goblet and determine how much liquid it can hold.

Checkpoint 2.7.26 A goblet is made up of a hemisphere mounted on top of a cylindrical base. The hemisphere has a radius of 28 m, and the cylindrical base has the same radius with a height of 45 m.

Answer the following:

1. Find the ***total external surface area*** of the goblet.

   Surface area =_______ m^2

2. Determine the ***maximum volume of liquid*** the goblet can hold.

   Volume =_______ m^3

Answer 1. 12848

Answer 2. 156875

Solution. ***Worked Solution***

The goblet consists of two parts:

- a ***cylindrical base***
- a ***hemisphere on top***

From the question, the dimensions are: Radius of the hemisphere and cylinder = 28 m Height of the cylindrical base = 45 m

1. ***Find the total external surface area of the goblet***

Step 1: Curved surface area of the cylinder

$$\begin{aligned}\text{CSA of cylinder} &= 2\pi rh\\ &= 2 \times \frac{22}{7} \times 28 \times 45\\ &= 7920.0\,\text{m}^2\end{aligned}$$

Step 2: Curved surface area of the hemisphere

$$\begin{aligned}\text{CSA of hemisphere} &= 2\pi r^2\\ &= 2 \times \frac{22}{7} \times (28)^2\\ &= 4928.0\,\text{m}^2\end{aligned}$$

Step 3: Total external surface area

$$\begin{aligned}\text{Total surface area} &= 7920.0 + 4928.0\\ &= 12848.0\,\text{m}^2\end{aligned}$$

2. ***Determine the maximum volume of liquid the goblet can hold***

Step 1: Volume of the cylinder

$$\begin{aligned}V_{\text{cylinder}} &= \pi r^2 h\\ &= \frac{22}{7} \times (28)^2 \times 45\\ &= 110880\,\text{m}^3\end{aligned}$$

Step 2: Volume of the hemisphere

$$\begin{aligned}V_{\text{hemisphere}} &= \frac{2}{3}\pi r^3\\ &= \frac{2}{3} \times \frac{22}{7} \times (28)^3\\ &= \frac{137984}{3}\,\text{m}^3\end{aligned}$$

Step 3: Total volume of goblet

$$\begin{aligned}V_{\text{total}} &= 110880 + \frac{137984}{3}\\ &= 156875\,\text{m}^3\end{aligned}$$

Checkpoint 2.7.27 This question contains interactive elements.

2.7.8 Surface area of a Frustum

Activity 2.7.11

Modelling a frustum using a paper cup or cone-shaped fruit juice glass and calculate its surface area.

Materials needed:

- Printable nets of a cone (to cut and create a frustum)

Rulers or measuring tape

Scissors

Tape or glue

Formula sheet

Worksheets for sketching and calculations

- A frustum is formed when the top part of a cone is cut off parallel to the base.
- Surface area includes:

  Curved surface area (side)

  Area of both circular bases
- Build the Frustum Model.

  Take the cone net and cut off the top part (smaller cone) parallel to the base.

  Assemble the remaining portion to form a frustum

  Alternatively, use actual paper/plastic cups and measure directly
- Label Dimensions

  Radius of the larger base (R)

  Radius of the smaller top base (r)

  Slant height (l) of the frustum

  (If not provided, measure the height and use the Pythagorean theorem)
- Calculate Surface Area
- Use the surface area formula:

  Total Surface Area $= \pi(R + r)\ell + \pi R^2 + \pi r^2$
- $\pi(R + r)$l: Curved surface

  πR^2: Area of bottom base

  πr^2: Area of top base
- Example: Given Top radius (r): 3 cm , Bottom radius (R): 5 cm and Slant height (l): 6 cm

$$\begin{aligned}\text{Surface Area } =&\pi(5\,\text{cm} + 3\,\text{cm})(6\,\text{cm}) + \pi(5\,\text{cm})^2 + \pi(3\,\text{cm})^2\\ =&3.14 \times (8\,\text{cm}) \times (6\,\text{cm}) + 3.14 \times 25\,\text{cm} + 3.14 \times 9\,\text{cm}\\ =&(3.14 \times\ 48\text{cm}) \times (3.14 \times\ 25\text{cm}) \times (3.14 \times\ 9\text{cm})\\ =&150.72\,\text{cm}^2 + 78.50\,\text{cm}^2 + 28.26\,\text{cm}^2\\ =&257.48\,\text{cm}^2\end{aligned}$$

- **Study Questions**.

  What would happen to the surface area if the top radius increased?

  Why is it necessary to measure the slant height, not the vertical height?

  Can you find any real-life objects shaped like a frustum?
- **Assignment**

  Design their own frustum cups with chosen dimensions.
- Given a full cone, how much surface area is “lost” when the top is cut off?

Key Takeaway

A frustum is a cone or pyramid is cut parallel to its base, removing the top portion. This results in a truncated shape with two parallel bases one smaller than the other e.g a lampshades, Truncated cones in engineering, buckets, Tunnels, Cooling towers in power plants etc

- Thereby a frustum is the portion of a cone (or pyramid) that remains after the top part is cut off parallel to the base.

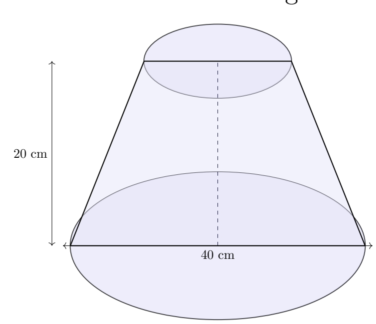
  Properties of a Frustum

- Two Circular Bases – A frustum has a larger base and a smaller base (both circular).
- Slant Height ℓ. The distance between the two bases along the side of the frustum.
- Height h – The vertical distance between the two bases.
- Curved Surface Area (CSA) – The side surface that connects the two bases.
- Total Surface Area (TSA) – The sum of the CSA and the areas of the two circular bases.
- **NOTE:** Volume, the space inside the frustum, is calculated using a formula derived from a full cone.

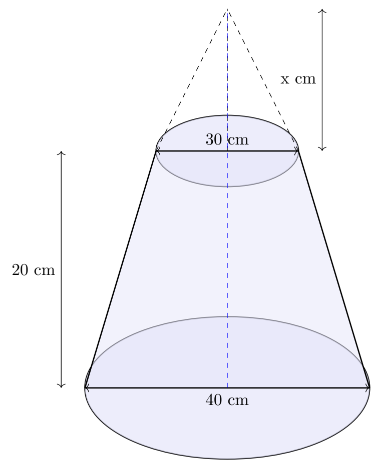
Types of frustums
Full cone
Important formulas to note;

1. Slant Height ℓ
$$\ell = \sqrt{(H+h)^2 + R^2}$$
2. Curved Surface Area (CSA)
$$CSA = \pi\, RL\, \text{and} \pi\, rl \text{ for the smaller cone}$$
3. 3. Total Surface Area (TSA)
$$\pi RL + \pi rl$$

Example 2.7.28 Find the surface area of the galvanized iron bucket below.



Solution. Complete the cone from which the bucket is made, by adding a smaller cone of height $x$cm.



From the cocept of similarity and enlargement;
$\frac{R}{r} = \frac{H}{h}$ and $\frac{H-h}{R-r} = \frac{h}{r}$

$$\begin{aligned}\frac{x}{15} =& \frac{x + 20\text{cm}}{20\text{cm}} \\ 20x =& 15x\text{cm} + 300\text{cm} \\ 300\text{cm} =& 20x - 15x \\ 300\text{cm} =& 5x \\ 60\text{cm} =& x\end{aligned}$$

Surface area of a frustrum = Area of curved surface of bigger cone - Area of curved surface of snaller cone
$\pi RL - \pi rl$

$$\begin{aligned}\text{Surface area (Large)} =& \frac{22}{7} \times 20\text{cm} \times \sqrt{80^2 + 20^2} \\ =& 5183.33\text{cm} \\ \text{Surface area (small)} =& \frac{22}{7} \times 15\text{cm} \times \sqrt{60^2 + 15^2} \\ =& 2915.62\text{cm} \\ \text{Differences in the Surface areas} =& 5183.33\text{cm} - 2915.62\text{cm} \\ =& 2267.71\text{cm}^2\end{aligned}$$

□

Exercise

1. A frustum of a square pyramid has:
Top square side length: 4 m
Bottom square side length: 6 m
Slant height (along one face): 5 m. Calculate the total surface area of the frustum.

2. A conical frustum has a bottom radius of 6 cm, no top (the top is flat), and a slant height of 10 cm.
i. Explain the difference between the curved surface area and the total surface area of a frustum.
ii. Find only the curved surface area of the frustum.

3. A frustum is formed by cutting a cone with a height of 24 cm into two parts. The smaller cone has a height of 9 cm. If the base radius of the original cone is 16 cm, calculate the total surface area of the frustum.

4. A conical frustum has a bottom radius of 6 cm, no top (the top is flat), and a slant height of 10 cm. Find only the curved surface area of the frustum.

5. If the curved surface area of a frustum is 330 cm^2, the top radius is 5 cm, and the bottom radius is 10 cm, find the slant height of the frustum.

6. A flower pot is shaped like a frustum of a cone. It has a top radius of 12 cm, a bottom radius of 8 cm, and a slant height of 10 cm.

Checkpoint 2.7.29 This question contains interactive elements.

Checkpoint 2.7.30 This question contains interactive elements.

2.7.9 Volume of Solids.

Activity 2.7.12 Materials Needed:

1. Transparent plastic containers shaped as: Cube or rectangular prism, Cylinder, Cone, Pyramid (square or triangular base) and Spheres (or hemispheres)
2. Sand, water, or rice (as filling material)
3. Measuring cup (with volume in millilitres or cm^3)
4. Large tray or basin (to catch spills)
5. Worksheet for recording observations and making predictions

- Take different empty containers and weigh them. Which shape do you think holds the most? or the least? Rank them from largest to smallest volume. Let them share and discuss predictions in small groups
- Fill each of them separately and Compare

  Start with the cylinder and cone with the same height. Fill the cone with sand/water and pour it into the cylinder
- How many cones fill the cylinder?

$$3\,\text{That is the cone is}\frac{1}{3}\text{a cylinder.}$$

- What is the volume of a cylinder? $V = \pi r^2$

  If a cone is $\frac{1}{3}$ a cylinder then it's volume will be; $\frac{1}{3}\pi r^2$
- Do the same with a pyramid and a matching prism (same base and height). How many pyramids fill the prism?

  3 pyramids fill tthe prism.
- What is the volume of a pyramid?

  $v = \frac{1}{2} \times \text{base area} \times h$
- Try filling the sphere into the cylinder (if you have a hemisphere: about 2 hemispheres = 1 sphere)
- Compare volumes visually and discuss the differences with your classmates.
- For cube or prism, measure directly with a ruler and calculate using $V = \text{length} \times \text{width} \times \text{height}$

- Measure the dimensions and use their measuring cups to check how much each container holds. They then calculate the actual volume using the formulas and compare their estimates and results.

Study Questions

1. What patterns do you notice between shapes that have the same base and height?
2. Why do you think cones and pyramids have a $\frac{1}{3}$ in their volume formula?
3. Which shapes are most efficient at holding volume?

Extended Activity

Design a container that holds exactly 500 cm³ using any shape. Estimate the volume of irregular solids using displacement.

Volume of a Cube

Volume is the geometric space occupied by an object, or the contents of an object. It is measured in cubic units.

Computation of a volume is achieved by multiplying the area of the base of the solid by the height of the solid

Let us proove this from the following examples.

Example 2.7.31 Find the volume of the following cube whose side is 5 cm.

Solution. Step 1: Find the area of the base

$$\begin{aligned}\text{Area of square} =& S^2\\ =& 5^2\\ =& 25\,\text{cm}^2\end{aligned}$$

Step 2: Multiply the area of the base by the height of the solid to find the volume

$$\begin{aligned}\text{volume} =& \text{areaof base} \times \text{height}\\ =& 25\,\text{cm}^2 \times 5\,\text{cm}\\ =& 125\,\text{cm}^3\end{aligned}$$

The volume of the cube is 125 $125\,\text{cm}^3$. □

Example 2.7.32 Finding the volume of a triangular prism.

Solution. Step 1: Find the area of the base

$$\begin{aligned}\textbf{area of triangle} =& \frac{1}{2}b \times h\\ =& \frac{1}{2} \times 9\,\text{cm} \times 12\,\text{cm}\\ =& 54\,\text{cm}^2\end{aligned}$$

Step 2: Multiply the area of the base by the height of the solid to find the volume

$$\begin{aligned}\text{Volume} =& \text{Base Area} \times \text{height}\\ =& \frac{1}{2}b \times h \times H\\ =& 54\,\text{cm}^2 \times 25\,\text{cm}\end{aligned}$$

$$=1350\,\text{cm}^3$$

The volume of the triangular prism is $1350\,\text{cm}^3$ □

Example 2.7.33 Find the volume of the following cylinder using $\pi = 3.142$. Leave your answer (correct to 2 decimal place):

Solution. Step 1: Find the area of the base

$$\begin{aligned} \text{area of circle } =& \pi r^2 \\ =& \pi(8)^2 \\ =& \pi 64\,\text{cm}^2 \\ =& 3,142 \times 64\,\text{cm}^2 \\ =& 201.088\,\text{cm}^2 \end{aligned}$$

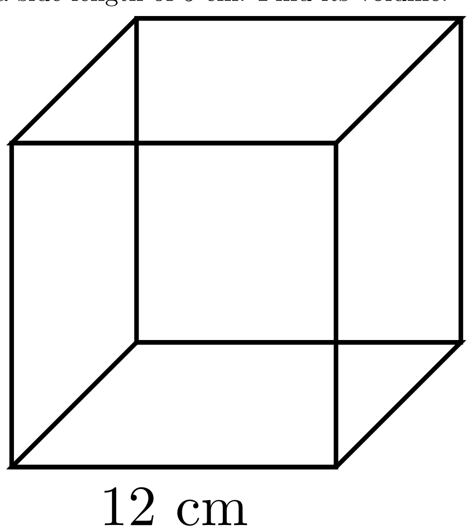
Step 2: Multiply the area of the base by the height of the solid to find the volume

$$\begin{aligned} \text{Volume } =& \text{Base area} \times \text{height} \\ =& \pi r^2 \times h \\ =& 201.088\,\text{cm}^2 \times 20\,\text{cm} \\ =& 4021.76\,\text{cm}^3 \end{aligned}$$

The volume of the cylinder is $4021.76\,\text{cm}^3$ □

Exercise

1. A cube has a side length of 5 cm. Find its volume.



2. The cube below has a volume of 343 cm³. Find the length of one side of the cube.

3. If the side length of a cube is doubled, by what factor does the volume increase?

4. A big cube is made by stacking 8 smaller identical cubes together. If the volume of each small cube is 27 cm^3, find the volume of the big cube.

5. (a) A cubical water tank has a side length of 3 m. How many liters of water can it hold when full?

(b) A cubical water tank is 2 m on each side. If it is filled completely, how many liters of water can it hold?

6. A sugar cube has a side length of 1 cm. If 1,000 sugar cubes are stacked together to form a larger cube, find the volume of the larger cube.

7. A freezer contains 200 ice cubes, each shaped like a cube with a side length of 4 cm. Find the total volume of ice inside the freezer.

8. A cubical shipping container has a side length of 2.5 m. Find the total volume of cargo space inside the container.

2.7.10 Volume of a Cuboid

Activity 2.7.13 Building with Cubes.

- Materials Needed:

  Small unit cubes e.g Match boxes,

  Rulers

  Grid paper

- Gather a set of small cubes.

  Build a cuboid using a 2 cubes at the width, 3 cubes at the longer side and 4 cubes for your height.

  Fill the entire cuboid completely until it is uniformly fitted.

  Count the total number of cubes used.

  Compare your answer with the calculated volume using the formula. **length** $\times$ **width** $\times$ **height**

- What happens if we change one dimension?

  Continue experimenting with different lengths, widths and heights.

Activity 2.7.14 Demonstrating how volume can be measured in real life situations.

- Materials Needed:

Water, sand or rice.

Rulers

Measuring cups

Transparent boxes or cubical containers of different sizes.

(You can take a cuboid-shaped Jerrycan cut the top to ensure it has a flat top and base)

Note book and pen for recording findings and computing sums.

- Measure and record the units for height, length and width of the container.

  Slowly fill the box(or container) with water, sand or rice

  Measure how many cups are needed to fill the box completely.

- What's the mass of the rice, sand or water that filled the cubical container? Convert to cubic Centimetres. Note down your answer.

  Calculate the volume using the formula $l \times w \times h$

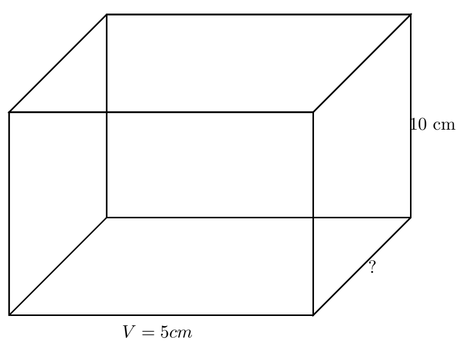
- Compare the actual measurement with their calculations.

NOTE: **Slight differences may occur due to gaps in sand, Spills while transfering or measurin**

Example 2.7.34 A cuboid has a length of 12 cm, a width of 8 cm, and a height of 5 cm. Find its volume. ☐

Exercise

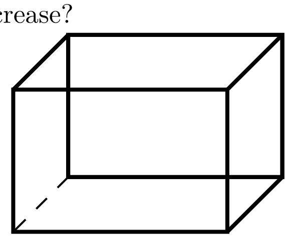
1. A cuboid has a volume of 600 cm³, a length of 10 cm, and a width of 5 cm. Find its height.

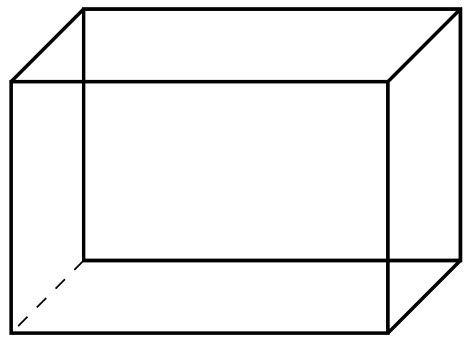


2. A rectangular water tank has a base of 2 m by 3 m and a height of 4 m. How many liters of water can it hold when full?

3. A shipping company uses boxes shaped like cuboids. Each box has a length of 40 cm, width of 30 cm, and height of 20 cm. If a warehouse has a storage space of 12 m³, how many such boxes can fit in the warehouse?

4.If the length, width, and height of a cuboid are all doubled, by what factor does the volume increase?

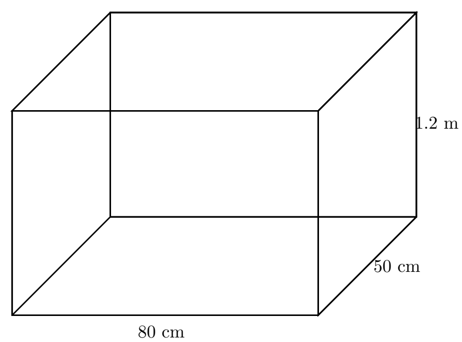




5. A brick has dimensions 20 cm by 10 cm by 5 cm. A wall is built using 500 such bricks, with no gaps between them. Find the total volume of bricks used in constructing the wall.

6. A wooden storage box has a length of 1.2 m, a width of 80 cm, and a height of 50 cm. Find the volume of the box in cubic meters.

(Hint: Convert all dimensions to meters before calculating.)



Checkpoint 2.7.35 This question contains interactive elements.

Checkpoint 2.7.36 The figure below represents a building block with dimensions.var board = JXG.JSXGraph.initBoard('box', { boundingbox: [-5, 15, 20, -5], axis: false, showNavigation: false, showCopyright: false }); // Front face (12 cm x 4 cm) var A = board.create('point', [2, 2], { visible: false ,fixed:true }); var B = board.create('point', [14, 2], { visible: false ,fixed:true }); var C = board.create('point', [14, 6], { visible: false ,fixed:true }); var D = board.create('point', [2, 6], { visible: false ,fixed:true }); // Back face (shifted for 3D effect) var E = board.create('point', [5, 5], { visible: false ,fixed:true }); var F = board.create('point', [17, 5], { visible: false ,fixed:true }); var G = board.create('point', [17, 9], { visible: false ,fixed:true }); var H = board.create('point', [5, 9], { visible: false ,fixed:true }); // Draw front and back faces var frontFace = board.create('polygon', [A, B, C, D], { borders: { strokeColor: 'blue' ,fixed:true } }); var backFace = board.create('polygon', [E, F, G, H], { borders: { strokeColor: 'blue' ,fixed:true } }); // Connect front and back faces with solid lines board.create('line', [A, E], { straightFirst: false, straightLast: false, strokeColor: 'blue' ,fixed:true }); board.create('line', [B, F], { straightFirst: false, straightLast: false, strokeColor: 'blue' ,fixed:true }); board.create('line', [C, G], { straightFirst: false, straightLast: false, strokeColor: 'blue' ,fixed:true }); board.create('line', [D, H], { straightFirst: false, straightLast: false, strokeColor: 'blue' ,fixed:true }); // Labels for dimensions board.create('text', [(A.X() + B.X()) / 2, A.Y() - 0.8, "33 cm"],{fixed:true }); board.create('text', [B.X() +2.3, (B.Y() + C.Y()) / 2, "39 cm"],{fixed:true }); board.create('text', [(G.X() + F.X()) / 2 +0.5, (G.Y() + F.Y()) / 2 + 0.9, "26 cm"],{fixed:true });

1. Calculate the volume of the block.

   Volume =_________ cm^3

2. Calculate the total surface area of the block.

   Surface area =_________ cm^2

3. If 10 such blocks are packed together to form a large cube-shaped structure, what is the total volume of the structure?

   Total volume =_________ cm^2

Answer 1. 33462

Answer 2. 6318

Answer 3. 334620

Solution. Given:

A building block has:

- Length $= 33\,\text{cm}$
- Width $= 39\,\text{cm}$
- Height $= 26\,\text{cm}$

There are 10 such identical blocks.

1. Find the volume of one block

   We use the formula for the volume of a cuboid:

   Volume = Length × Width × Height

   Where:

   $l = 33\,\text{cm}$

   $w = 39\,\text{cm}$

   $h = 26\,\text{cm}$

   Substitute the given values:

$$\begin{aligned}\text{Volume} &= 33 \times 39 \times 26 \\ &= 33462\,\text{cm}^3\end{aligned}$$

   Therefore,

   The volume of the building block $= 33462\,\text{cm}^3$

2. Find the total surface area of one block

   The surface area of a cuboid is calculated using:

   Surface Area $= 2(lw + wh + hl)$

$$\begin{aligned}\text{Surface Area} &= 2\left((33 \times 39) + (39 \times 26) + (33 \times 33)\right) \\ &= 2(1287 + 1014 + 858) \\ &= 2 \times 3159 \\ &= 6318\,\text{cm}^2\end{aligned}$$

   Therefore,

   The surface area of the building block is $6318\,\text{cm}^2$

3. Find the total volume of all 10 blocks

   We already found the volume of one block is: $= 33462\,\text{cm}^3$

   Now multiply by the number of blocks:

$$\text{Total Volume} = 33462 \times 10$$
$$= 334620\,\text{cm}^3$$

   Therefore,

   Total volume of $10\,\text{blocks} = 334620\,\text{cm}^3$

2.7.11 Volume of a Triangular Prism

Activity 2.7.15 Work together in small groups of 5 to:

Discuss ideas

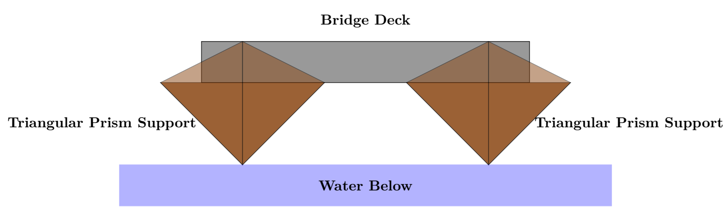
Distribute tasks (e.g., measuring, cutting, assembling)

Share calculations

- Building a Bridge

  V $=$ Base Area $\times$ Height

  where the base area is given by $\frac{1}{2} \times$ Base $\times$ height of the triangle.



Skills to be Developed: Measurement, visualization, real-world connection

- Materials needed:

  Cardboard or wooden sticks

  triangular prisms

  Using the cardboards or wooden sticks construct a bridge model with the triangular prism acting as supports as shown alongside.

  Have them measure the base, height of the triangle, and length of the prism.

- Use the formula to calculate the volume of the prism-shaped supports.

- Compare different bridge designs and discuss which structure is the strongest.

  Why Choose the "Build a Bridge" Activity?

  Bridges are a perfect example of triangular prisms in engineering. Many bridges use triangular trusses because they:

  Distribute weight evenly

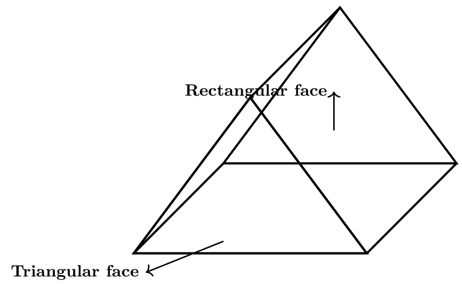
  Provide structural stability

  Are used in real-life construction

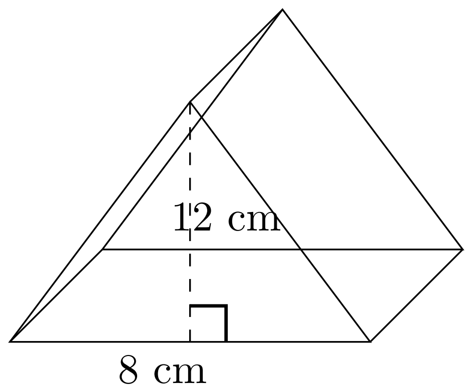
  By building a model bridge, students get to see, touch and manipulate triangular prisms, helping them connect abstract mathematical concepts to real-world engineering.

More Than Just Math!

The **"Build a Bridge"**activity isn't just about calculating volume—it's about seeing math in action!



Example 2.7.37 Find the Surface area of the triangular prism below.



Solution. Step 1: Find the area of the base.

$$\begin{aligned}\text{area of a triangle} =&\frac{1}{2}b\times h\\ =&\frac{1}{2}\times 8\text{cm}\times 12\,\text{cm}\\ =&48\,\text{cm}^2\end{aligned}$$

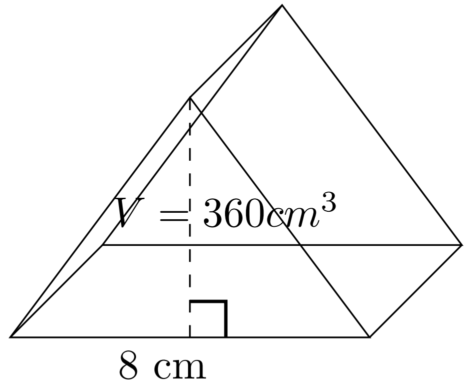
Step 2: Multiply the area of the base by the height of the solid to find the volume

$$\begin{aligned}\text{volume} =&\text{area of base}\times\text{height}\\ =&\frac{1}{2}b\times h\times H\\ =&48\,\text{cm}^2\times 12\,\text{cm}\\ =&576\,\text{cm}^3\end{aligned}$$

The volume of the triangular prism is $576\,\text{cm}^3$ □

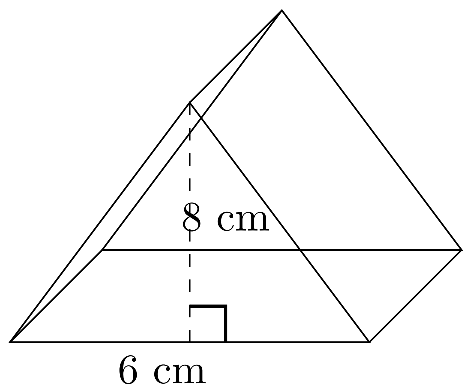
Exercise

1. A triangular prism has a volume of 360 cm^3. The base of the triangular cross-section is 10 cm, and the height of the triangle is 9 cm. Find the length of the prism.



2. A water trough is in the shape of a triangular prism. The triangular cross-section has a base of 10 cm and a height of 12 cm. The trough is 2 meters long. How much water can it hold in liters? (Hint: 1 cm^3 = 1 mL, and 1,000 mL = 1 L)

3. A triangular prism has a triangular base with a base length of 8 cm and a height of 6 cm. Find the total volume of the solid.

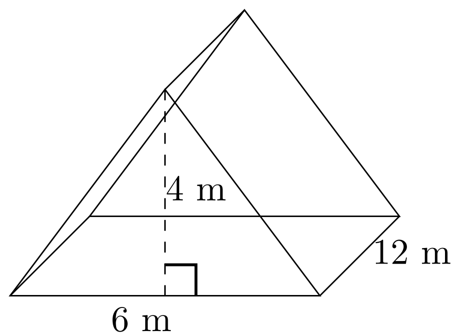


4. A storage container is shaped like two triangular prisms joined together along their rectangular faces. Each triangular prism has a base of 5 cm, a height of 4 cm, and a length of 20 cm. Find the total volume of the container.

5. A house-shaped block consists of a rectangular prism 6 cm by 8 cm by 12 cm with a triangular prism (with a base of 8 cm and a height of 6 cm) attached on top. The figures below shows the blocks whey they are divided into two equal halves. Find the total volume of the solid.

6. A company manufactures Toblerone-shaped chocolate bars, which are shaped like triangular prisms. Each bar has a triangular cross-section with a base of 5 cm and a height of 4 cm, and the length of the chocolate bar is 30 cm. Find the volume of a single chocolate bar. If 100 bars are packed into a box, what is the total volume of chocolate in the box?

7. An architect designs a triangular prism-shaped roof for a house. The triangular cross-section has a base of 6 m and a height of 4 m. The length of the roof is 12 m. Calculate the total volume of the roof structure.



2.7.12 Volume of a Pyramid

Activity 2.7.16 "Pyramid City"

- Materials

  Pictures of famous pyramids (e.g., Egyptian Pyramids, Mayan Pyramids)

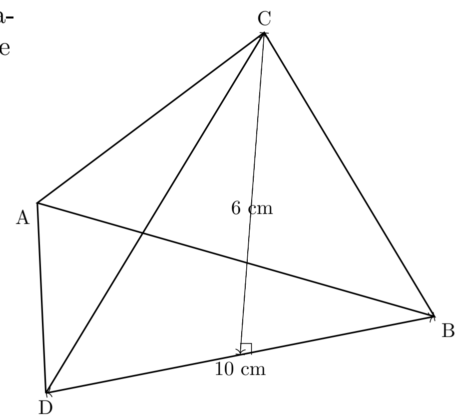
  Measuring tape or rulers (for estimating dimensions)

  A small model or LEGO pyramid

A pyramid has a polygonal base and triangular faces that meet at the apex.

Example 2.7.38

Find the volume of a square pyramid with a height of 6 cm and a side length of 10cm.



Solution. Step 1: Select the correct formula and substitute the given values.

We are given b = 10 and H = 6, therefore

$$\begin{aligned} V =& \frac{1}{3} \times \text{base Area} \\ \text{Base Area} =& (10\text{cm} \times 10\text{cm}) \\ =& \frac{1}{3} \times (10 \times 10)\text{cm}^2 \times 6\text{cm} \\ =& 100\text{cm}^2 \times 2\text{cm} \\ =& 200\text{cm}^3 \end{aligned}$$

The volume of the square pyramid is 200cm^3. □

Example 2.7.39 A square pyramid has a base of 6 cm × 6 cm and a height of 9 cm. Find it's volume.

Solution.

$$\begin{aligned} V =& \frac{1}{3} \times \text{base Area} \times h \\ \text{Base Area} =& (6\text{cm} \times 6\,\text{cm}) \\ =& \frac{1}{3} \times (6 \times 6)\,\text{cm} \times 9\,\text{cm} \\ =& \frac{1}{3} \times 36\,\text{cm}^2 \times 9\,\text{cm} \\ =& 108\,\text{cm}^3 \end{aligned}$$

□

Example 2.7.40 A triangular pyramid has a base of 5 cm × 8 cm and a height of 10 cm.

Solution.

$$\begin{aligned} V =& \frac{1}{3} \times \text{base Area} \times h \\ \text{Base Area} =& (\frac{1}{2} \times 5\,\text{cm} \times 8\,\text{cm}) \\ =& \frac{1}{3} \times (\frac{1}{2} \times 5\,\text{cm} \times 8\,\text{cm}) \times 10\,\text{cm} \\ =& \frac{1}{3} \times 20\,\text{cm}^2 \times 10\,\text{cm} \\ =& 66.67\,\text{cm}^3 \end{aligned}$$

□

Example 2.7.41 A pyramid has a rectangular base of 4 m by 6 m and a height of 12 m.

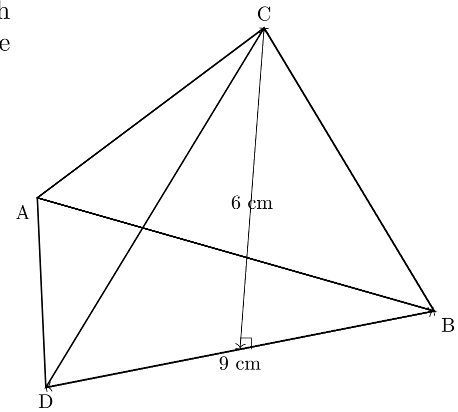
Solution.

$$\begin{aligned} V =& \frac{1}{3} \times \text{base Area} \times h \\ \text{Base Area} =& (\times 4\,\text{m} \times 6\,\text{m}) \\ =& \frac{1}{3} \times \times (4\,\text{m} \times 6\,\text{m}) \times 12 \\ =& \frac{1}{3} \times 24\,\text{m}^2 \times 12\,\text{m} \\ =& 96\,\text{m}^3 \end{aligned}$$

□

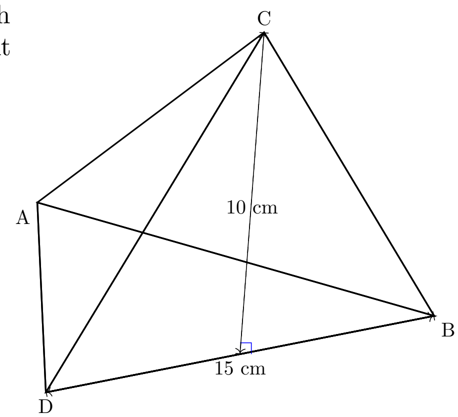
Exercise

1. A pyramid has a square base with a side length of 6 cm. The height of the pyramid is 9 cm.

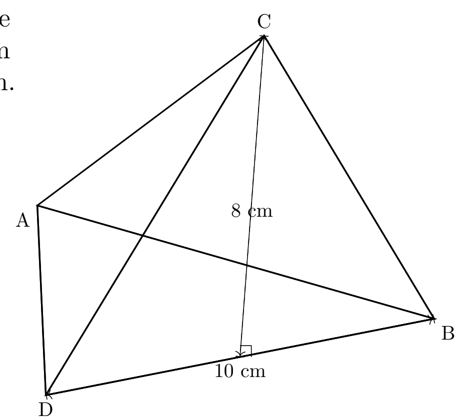


2. A pyramid-shaped tent has a rectangular base of 8 m by 6 m and a height of 5 m. Find the volume of air inside the tent.

3. A pyramid has a square base with each side measuring 10 cm. The height of the pyramid is 15 cm.



4. A pyramid has a triangular base where the base of the triangle is 8 cm and the height of the triangle is 6 cm. The height of the pyramid is 10 cm.



5. A decorative garden pyramid has a square base with each side measuring 4 m. The height of the pyramid is 3 m.

Checkpoint 2.7.42 The figure below shows a pyramid with a triangular base. The base of the triangle measures 18 m, the height of the triangle is 3 m, and the vertical height of the pyramid is 28 m. Find the volume of the pyramid.

Volume =_______ m^3

Answer. 252

Solution. ***Given:***

Base of triangle, $b = 18\,\text{m}$ Height of triangle, $h = 3\,\text{m}$ Vertical height of pyramid, $H = 28\,\text{m}$

1. Compute the area of the triangular base:

$$\begin{aligned}\text{Base area} &= \frac{1}{2} \times \text{base} \times \text{height of triangle}\\ &= \frac{1}{2} \times 18 \times 3\\ &= 27\,\text{m}^2\end{aligned}$$

2. Compute the volume of the pyramid using $V = \frac{1}{3} \times \text{Base area} \times$ vertical height:

$$\begin{aligned}V &= \frac{1}{3} \times 27 \times 28\\ &= 252\,\text{m}^3\end{aligned}$$

Therefore, The volume of the pyramid is $252\,\text{m}^3$

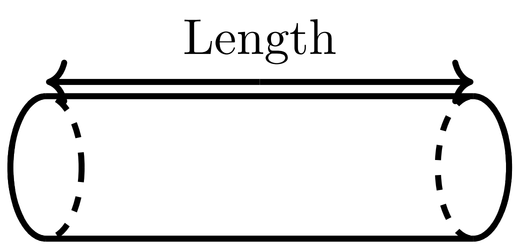
Checkpoint 2.7.43 This question contains interactive elements.

2.7.13 Volume of a Cylinder

Activity 2.7.17 Work in groups
What you require: A pair of scissor and a piece of paper.

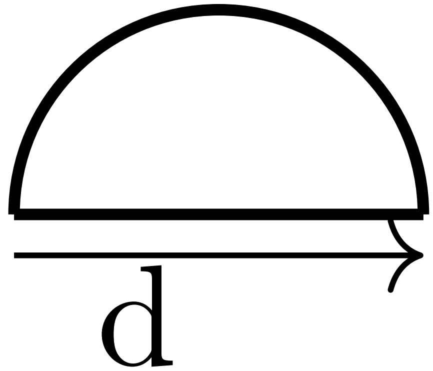
(i). Make a paper made of a cylinder.

(ii). Measure and record the the length of the model.



(iii). Open the model as illustrated below.

(iv). Fold one circular end into two equal parts as shown;



(v). Measure and recod the diameter.

(vi). Calculate the circumference of the circular end.

(vii). Rotate the width of the rectangular part with the diameter of the circular part.

(viii). Find the area of the rectangular part.

(ix). Find the area of the circular ends.

(x). Find the surface area of the cylinder.

(xi). Discuss and share your answer with other groups

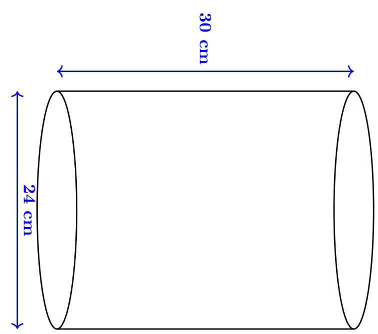
Key point

The total surface area of a cylinder of radius r and height h, is given by the sum of the areas of the two circular faces and the curves face. Thus,

$$\textbf{Total surface area of a cylinder} = 2\pi r^2 + 2\pi r h$$
$$= 2\pi r(r + h)$$

Note: A cylinder is always considered closed, unless it is specified that it is open.

Example 2.7.44 Calculate the surface area of the cylinder shown.



Solution. Given that, $h = 30\,\text{cm}$ and $d = 24\,\text{cm}$

We first get the radius of the cylinder which is given by,

$$r = \frac{d}{2}$$

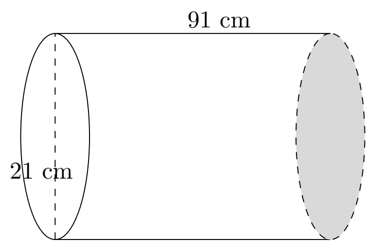
$$= \frac{24}{2}$$
$$= 12\,\text{cm}$$

Therefore,

$$\begin{aligned}\text{Surface area} &= 2\pi r^2 + 2\pi rh\\ &= 2\pi r(r+h)\\ &= 2\times\pi\times 12\,\text{cm}\,(12\,\text{cm} + 30\,\text{cm})\\ &= 2\times\pi\times 12\,\text{cm}\ \times 42\,\text{cm}\\ &= 2\times 22\times 12\times 6\\ &= 3\,168\,\text{cm}^2\end{aligned}$$

The surface area of the above cylinder is $3\,168\,\text{cm}^2$ □

Example 2.7.45 The figure below shows a cylinder. Calculate the surface area of the cylinder to two decimal places.(**Use** $\pi = \frac{22}{7}$)



Solution. Given that, $h = 91\,\text{cm}$ and $d = 21\,\text{cm}$
We first get the radius of the cylinder which is given by,

$$\begin{aligned}r &= \frac{d}{2}\\ &= \frac{21}{2}\\ &= 10.5\,\text{cm}\end{aligned}$$

Therefore,

$$\begin{aligned}\text{Surface area} &= 2\pi r(r+h)\\ &= 2\times\pi\times 10.5\,\text{cm}(10.5\,\text{cm} + 91\,\text{cm})\\ &= 2\times\pi\times 10.5\,\text{cm}\times 101.5\,\text{cm}\\ &= 6\,699.00\,\text{cm}^2\end{aligned}$$

The surface area of the cylinder= $6\,699.00\,\text{cm}^2$ □

Example 2.7.46 Calculate the surface area of unsharpened circular pencil in the shape of a cylinder whose radius is $0.2\,\text{m}$ and height is height is $1.4\,\text{m}$. (**Use** $\pi = 3.142$)

Solution. Given that, $h = 01.4\,\text{m}$ and $r = 0.2\,\text{m}$
Therefore,

$$\text{Surface area} = 2\pi r(r+h)$$

$$
\begin{aligned}
&= 2 \times 3.142 \times 0.2\,\text{m}(0.2\,\text{m} + 1.4\,\text{m}) \\
&= 2 \times 3.142 \times 0.2\,\text{m} \times 1.6\,\text{m} \\
&= 2.01088\,\text{m}^2
\end{aligned}
$$

Therefore, surface area = $2.01088\,\text{m}^2$ □

Activity 2.7.18 Work in groups

What you require: A cylindrical container without a lid, a pair of scissores and a piece of paper.

1. Trace the bottom face of the cylindrical container on a piece of paper and cut out the shape.
2. (a) Entirely cover the curved surfaceof the container with a piece of paper.
   (b) Cut off any parts of the piece of paper that extend beyond the curved surface. Also, ensure the piece of paper does not overlap.
3. Calculate the area of each of the two cutouts.
4. Work out the total area of all the cutouts. What does the area represent?
5. Discuss how to calculate the surface area of an open cylinder and share your findings with other groups in your class.

Key Takeaway

An open cylinder, is a cylinder where the top is open. It means that you will only have one circle instade of two.

Therefore, to calculate the surface area of the cylinder you add the area of the curved surface and the circle.

An open cylinder has a curved surface and one circular face.

Therefore,

$$
\begin{aligned}
\textbf{Surface area of an open cylinder} &= \pi r^2 + 2\pi r h \\
&= \pi r(r + 2h)
\end{aligned}
$$

Example 2.7.47 An open cylindrical cotainer has a height of 12 cm and a diameter of 2 cm. What is the surface area of the outer surfaces of the container? ($\pi = 3.142$)

Solution. You are given the height and diameter of the container to be $h = 12\,\text{cm}$ and $d = 2\,\text{cm}$

You first identify the radius of the container that is,

$$
\begin{aligned}
r &= \frac{2}{2} \\
&= 1\,\text{cm}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\text{surface area} &= \pi r(r + 2h) \\
&= 3.1421(1 + 2 \times 12) \\
&= 3.1421(1 + 24)\,\text{cm} \\
&= 3.142 \times 25\,\text{cm} \\
&= 78.55\,\text{cm}^2
\end{aligned}
$$

Therefore, surface area of the outer container is $= 78.55\,m^2$. □

Example 2.7.48 Ekadeli filled an open cylindrical bucket with water. The internal diameter of the bucket was 32.4 cm and the internal height was 35 cm. Calculate the area of the bucket that was in contact with the water. Write the answer correct to **1 decimal place**. (**Use** $\pi = \frac{22}{7}$)

Solution. The open cylindrical bucket has a diameter and height of $d = 32.4\,\text{cm}$ and $h = 35\,\text{cm}$.

You first calculate the radius of the cylindrical bucket which is,

$$\begin{aligned} r =& \frac{d}{2} \\ =& \frac{32.4}{2} \\ =& 16.2\,\text{cm} \end{aligned}$$

Therefore,

$$\begin{aligned} \text{Surface area} =& \pi r(r + 2h) \\ =& \frac{22}{7} \times 16.2\,\text{cm} \times (16.2\,\text{cm} + 2 \times 35\,\text{cm}) \\ =& \frac{22}{7} \times 16.2\,\text{cm} \times (16.2\,\text{cm} + 70\,\text{cm}) \\ =& \frac{22}{7} \times 16.2\,\text{cm} \times 86.2\,\text{cm} \\ =& 4388.81142857\,\text{cm}^2 \end{aligned}$$

The surface area $= 4388.8\,\text{cm}^2$ □

Example 2.7.49 The surface area of an open jar is $594\,\text{cm}^2$. The radius of the jar is 7 cm. calculate the height of the jar. (**Use** $\pi = 3.142$).

Solution. You are given,

Surface are $= 594\,\text{cm}^2$

$r = 7\,cm$

$h = ?$

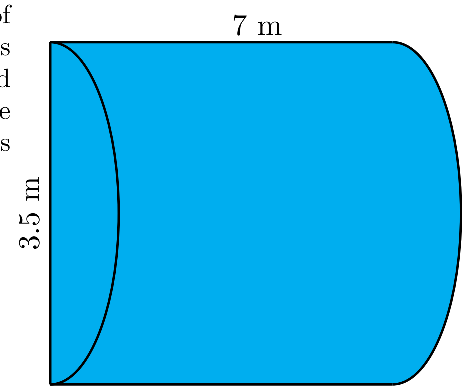
To find the height of the jar we substitute the above values in the formula below.

$$\begin{aligned} \text{Surface area} =& \pi r(r + 2h) \\ 594\,\text{cm}^2 =& \frac{22}{7} \times 7\,\text{cm}(7\,\text{cm} + 2 \times h) \\ \frac{594}{22} =& 7\,\text{cm} + 2 \times h \\ (27 - 7)\,\text{cm} =& 2 \times h \\ 20\,\text{cm} =& 2 \times h \\ \frac{20}{2}\,\text{cm} =& h \\ h =& 10\,\text{cm} \end{aligned}$$

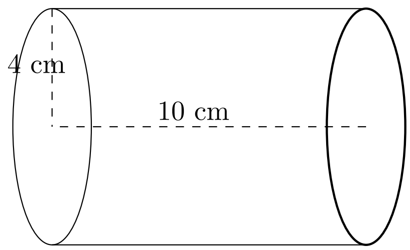
The height of the jar $= 10\,\text{cm}$ □

Extended Exercise

The figure alongside shows the roof of a motorbike shade. The roof is painted on the outer curved surface and the two semi-circle faces. Calculate the surface area of the part of roof that is painted.



Example 2.7.50 The cylinder below has a radius of 4 cm and a height of 10 cm. Use $\pi = 3.142$



$$\begin{aligned}V =& \text{Base Area } \times h\\ V =& \pi r^2 \times h\\ =& 3.142 \times (4)^2 \,\text{cm} \times 10 \,\text{cm}\\ =& 502.65 \,\text{cm}^3\end{aligned}$$

□

Example 2.7.51 A pipe has a radius of 3 cm and a height of 15 cm. Find the volume of the pipe.

Solution.

$$\begin{aligned}V =& \text{Base Area} \times h\\ V =& \pi r^2 \times h\\ =& 3.142 \times (3)^2 \,\text{cm} \times 15 \,\text{cm}\\ =& 424.12 \,\text{cm}^3\end{aligned}$$

□

Example 2.7.52 Mueni's water tank has a radius of 7 m and a height of 20 m. Find the capacity of the water tank in litres.

Solution.

$$\begin{aligned}V =& \text{Base Area} \times h\\ V =& \pi r^2 \times h\\ =& \frac{22}{7} \times (7)^2 \,\text{m} \times 20 \,\text{m}\\ =& 3080 \,\text{m}^3\end{aligned}$$

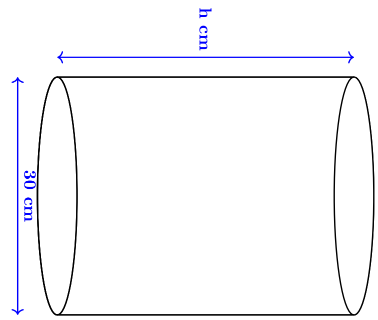
To convert from m^3 to litres we multiply by $1000\,\mathrm{m}^3$

$$\begin{aligned} 1\text{litre} =& 1000\,\mathrm{m}^3 \\ =& 3080\,\mathrm{m}^3 \times 1000\,\mathrm{m}^3 \\ =& 3,080,000 \ \text{ litres} \end{aligned}$$

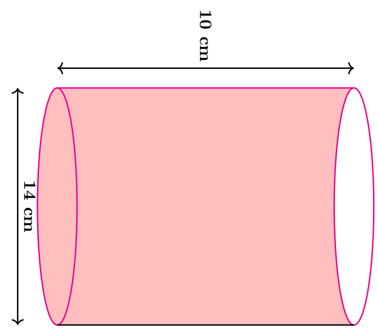
□

Exercise

1. A cylinder has a volume of 1,570 cm³ and a diameter of 30 cm. Find its height.Use ($\pi = 3.14$)



2. Alongside is a cross-sectional view of an open cylinder that has a radius of 7 cm and a height of 10 cm. Find its volume.



3. A hollow cylindrical pipe has an outer radius of 8 cm, an inner radius of 6 cm, and a length of 100 cm. Find the volume of material used to make the pipe.

4. A cylindrical water tank has a radius of 2 m and a height of 5 m. How many liters of water can it hold when full?

5. If the radius of a cylinder is doubled while keeping the height the same, by what factor does the volume increase?

Checkpoint 2.7.53 This question contains interactive elements.

Checkpoint 2.7.54 A hollow cylindrical pipe has an outer radius of 50 m, an inner radius of 5 m and a length of 100 m. Calculate the volume of material used to make the pipe.

Volume of the pipe =_______ m^3

Answer. 777645

Solution. Given:

Outer radius: 50 m

Inner radius: 5 m

Length of the pipe: 100 m

Step 1: Compute the volume of the outer cylinder

$$\begin{aligned} V_{\text{Outer}} &= \pi R^2 h \\ &= 3.142 \times 50^2 \times 100 \\ &= 785500.0\,\text{m}^3 \end{aligned}$$

Step 2: Compute the volume of the inner cylinder

$$\begin{aligned} V_{\text{Inner}} &= \pi r^2 h \\ &= 3.142 \times 5^2 \times 100 \\ &= 7855.0\,\text{m}^3 \end{aligned}$$

Step 3: Subtract inner volume from outer volume to get the hollow cylinder volume

$$\begin{aligned} V &= V_{\text{Outer}} - V_{\text{Inner}} \\ &= 785500.0 - 7855.0 \\ &= 777645\,\text{m}^3 \end{aligned}$$

2.7.14 Volume of a Sphere

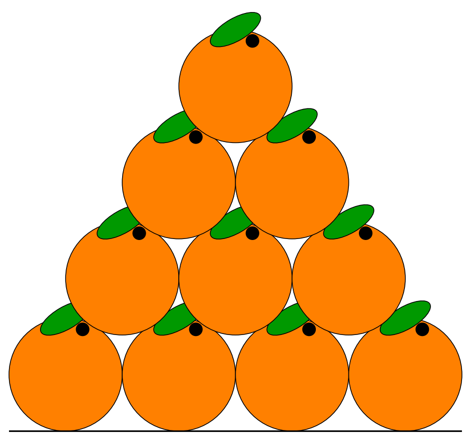
Activity 2.7.19 Orange Peeling Experiment

- Materials Needed:

  An orange (or any round fruit)

  A knife

  A flat surface



Pyramidal Stack of Oranges

- Cut the orange in half and carefully peel the skin off in small sections.
- Try flattening the peels and arrange them to see how they approximate a circle's area.

  When you peel an orange and flatten the pieces, you can see that the peels cover a large area.

  This helps visualize why increasing the radius increases the overall amount of space the fruit takes up (its volume).

- Cutting the Orange into Sections.

  If you cut an orange in half, you can see its cross-section.

  If you keep slicing it into smaller spheres, their individual radii determine their volumes.

- Why do you think oranges or tomatoes or apples etc. are stacking in pyramidal stacks in the market?

  Oranges in a fruit market are often packed in pyramidal stacks because spheres fit together efficiently.

  The larger the radius, the more space each orange occupies, which directly affects storage and packaging.

- Key takeaways

  The radius is the most important factor in determining the volume of a sphere.

  If the radius doubles, the volume increases by $2^3 = 8$ times!

  This explains why a slightly bigger orange holds significantly more juice compared to a smaller one.

Key Takeaway
A sphere is a perfectly round object.

Example 2.7.55 A sphere has a radius of 6 cm. Find its volume.

$$\begin{aligned}\text{V} =&\frac{4}{3} \times \pi r^3\\ =&3.142 \times \frac{4}{3} \times (6)^3\\ =&\frac{864}{3} \times 3.142\\ =&904.9\,\text{cm}^3\end{aligned}$$

□

Example 2.7.56 A football has a radius of 9 cm. What is the volume of the ball?

Solution.

$$\begin{aligned}\text{V} =&\frac{4}{3} \times \pi r^3\\ =&3.142 \times \pi\frac{4}{3} \times (9\,\text{cm})^3\\ =&3054.02\,\text{cm}^3\end{aligned}$$

□

Example 2.7.57 A planet has a radius of 1000 km. What's it's volume?

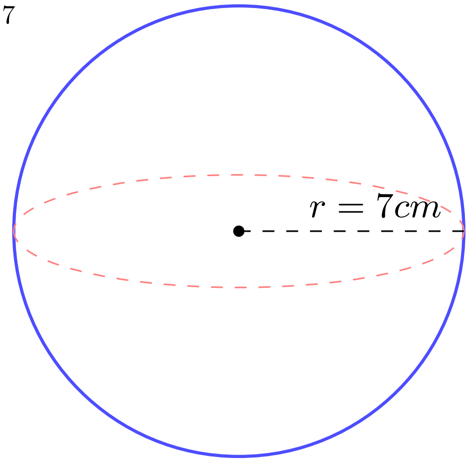
Solution.

$$\begin{aligned}\text{V} =&\frac{4}{3} \times r^3\\ =&3.142 \times \frac{4}{3} \times (1000)^3\\ =&4,189,333,3334.19 = \ 10^9\,\text{km}^3\end{aligned}$$

□

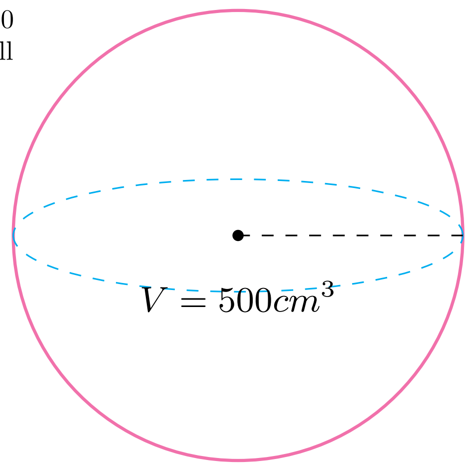
Exercise

1. A solid sphere has a radius of 7 cm. Find its volume.



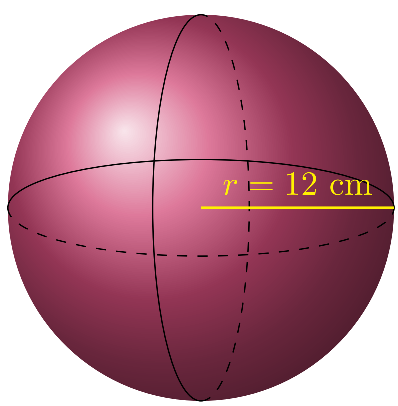
2. A bowl is in the shape of a hemisphere with a diameter of 12 cm. Find the volume of the bowl.

3. A sphere has a volume of 500 cm³. If the radius is doubled, what will be the new volume?



4. A raindrop is modeled as a sphere with a radius of 0.2 cm. If a storm produces 1,000,000 raindrops, what is the total volume of water in liters?

5 A basketball has a radius of 12 cm, while a tennis ball has a radius of 4 cm. How many tennis balls can fit inside the basketball, assuming no empty space?



Sphere with radius $r = 12cm$

Checkpoint 2.7.58 This question contains interactive elements.

Checkpoint 2.7.59 A spherical container has a diameter of 26 cm and is $\frac{4}{3}$ full of water. The water is emptied into a cylindrical container of diameter

14 cm. Find the depth of the water in the cylindrical container.
The depth of the water =_______ cm

Answer. 79.7097505669

Solution. ***Step 1: Write down the given information***
Diameter of the sphere = 26 cm
Diameter of the cylinder = 14 cm
Fraction of the sphere filled with water = $\frac{4}{3}$

Step 2: Find the radii of the sphere and the cylinder

$$\begin{aligned} r_{\text{sphere}} &= \frac{\text{Diameter}}{2} \\ &= \frac{26}{2} \\ &= 13\,\text{cm} \end{aligned}$$

$$\begin{aligned} r_{\text{cylinder}} &= \frac{\text{Diameter}}{2} \\ &= \frac{14}{2} \\ &= 7\,\text{cm} \end{aligned}$$

Step 3: Find the volume of water in the sphere

$$\begin{aligned} V_{\text{sphere}} &= \frac{4}{3} \times \frac{4}{3}\pi r^3 \\ &= \frac{4}{3} \times \frac{4}{3} \times \frac{22}{7} \times 13^3 \\ &= \frac{16}{9} \times \frac{22}{7} \times R_s \\ &= \frac{16}{9} \times v \\ &= \frac{773344}{63} \\ &= v_{sphere}\,\text{cm}^3 \end{aligned}$$

Step 4: Write the formula for the volume of water in the cylinder

$$\begin{aligned} V_{\text{cylinder}} &= \pi r^2 h \\ &= \frac{22}{7} \times 7^2 \times h \end{aligned}$$

Step 5: Equate the two volumes to find the depth of water

$$\begin{aligned} V_{sphere} &= V_{cylinder} \\ v_{sphere} &= \frac{22}{7} \times 7^2 \times h \\ h &= \frac{v_{sphere}}{\frac{22}{7} \times 7^2} \\ &= \frac{v_{sphere}}{Cy} \end{aligned}$$

$$= \frac{35152}{441}$$
$$= 79.7097505669\,\text{cm}$$

Therefore,

$$\text{Depth of the water} = 79.7097505669\,\text{cm}$$

2.7.15 Volume of a Cone

Activity 2.7.20 Constructing a cone.

1. Materials Needed

   Sheets of paper or cardboard

   Scissors, glue/tape, and rulers

   A cylinder (e.g., cup or bottle) for comparison

   - "How can we turn this into a cone?"
     Take a piece of paper and **cut a circle** any radius.
   - **Cut out a sector and roll** the remaining part into a cone shape.
   - Measure the radius and height of their cones.
   - Calculate the volume using the formula .

   ■ Equally, we can try this activity using; Empty Ice cream cones, A cylindrical cup of the **same height** and **base** as the cone and water.
   Steps:

   ■ Fill the cone with water and pour it into the cylinder severally until the cylinder is full.

   ■ How many cones of rice will fill the cylinder?

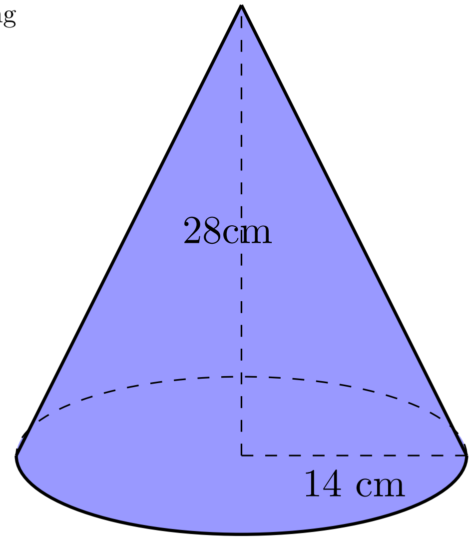
   ■ **Notice that it takes exactly 3 full cones to fill the cylinder**. This is why the formula includes $\mathbf{\frac{1}{3}}$!
   Mathematical Insight
   This shows the formula:

$$V_{\text{cone}} = \frac{1}{3}V_{\text{cylinder}} = \frac{1}{3}\pi r^2 h$$

   The cone is one-third of the volume of a cylinder with the same base and height.

Example 2.7.60

Find the volume of the following cone (correct to 1 decimal place):



Solution. Step 1: Find the area of the base .

$$\begin{aligned}\text{Area of a Circle} =& \pi r^2\\ =& \frac{22}{7} \times 14\,\text{cm} \times 14\,\text{cm}\\ =& 616 cm^2\end{aligned}$$

Step 2: Calculate the volume

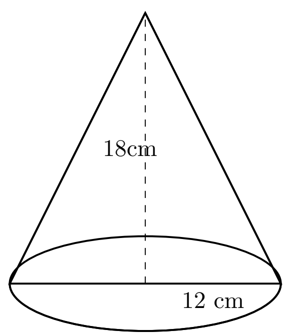
$$\begin{aligned}\text{V} =& \frac{1}{3} \times \pi r^2 \times H\\ =& 616\,\text{cm}^2 \times 28\,\text{cm}\end{aligned}$$

$$= 17,248\,\text{cm}^3$$

□

Exercise

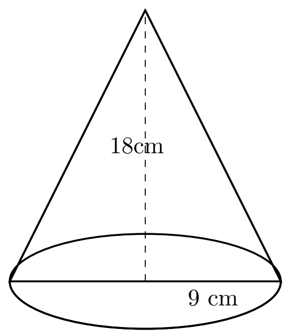
1. A cone has a radius of 12 cm and a height of 18 cm. Calculate the volume of the cone.



2. An ice cream cone has a radius of 3 cm and a height of 8 cm. Estimate how much ice cream it can hold

3. Two cones have the same height of 84 cm but different radii. The first cone has a radius of 14 cm, and the second cone has a radius of 42 cm. Calculate and compare the volumes of the two cones. Which one has a larger volume?

4. A cone-shaped funnel has a radius of 9 cm and a height of 18 cm.How much water can the funnel hold? (Leave your answer in cubic meters.)



5. A cone has an outer radius of 7 cm and an inner radius of 5 cm. The height of the cone is 12 cm.Calculate the volume of the hollow cone.

Checkpoint 2.7.61 A cone has a perpendicular height of 20 cm and a slant height of 25 cm. Take $\pi = \frac{22}{7}$.

Determine the following:

1. the radius of the cone

   Radius =_______ cm

2. the volume of the cone to the nearest whole numbers

   Volume =_______ cm^3

Answer 1. 15

Answer 2. 4714

Solution. The cone has a perpendicular height of 20 cm and a slant height of 25 cm.

Step 1: Determine the radius of the cone

For a cone, the radius r, the perpendicular height h, and the slant height l form a right-angled triangle.

$$l^2 = r^2 + h^2$$

Substituting the given values:

$$\begin{aligned}(25)^2 &= r^2 + (20)^2\\ r^2 &= (25)^2 - (20)^2\\ r^2 &= 25^2 - 20^2\\ r &= \sqrt{25^2 - 20^2}\\ &= 15\,\text{cm}\end{aligned}$$

Step 2: Find the volume of the cone

The formula for the volume of a cone is:

$$\begin{aligned}V &= \frac{1}{3} \times \text{base area} \times \text{height}\\ \text{Base area} &= \pi r^2\\ &= \frac{22}{7} \times (15)^2\\ &= \frac{4950}{7}\,\text{cm}^2\end{aligned}$$

Therefore,

$$\begin{aligned} V &= \frac{1}{3} \times \frac{4950}{7} \times 20 \\ &= \frac{1}{3} \times \frac{99000}{7} \\ &= \frac{33000}{7} \\ &= 4714 \text{ cm}^3 \end{aligned}$$

Rounded to the nearest whole number:

$$V = 4714 \text{ cm}^3$$

Therefore:

1. The radius of the cone is 15 cm.
2. The volume of the cone is 4714 $\boldsymbol{cm}^3$.

Checkpoint 2.7.62 This question contains interactive elements.

2.7.16 Volume of a Frustum

Activity 2.7.21 Frustum Volume Lab

Use a bucket-shaped container to model a frustum and measure how much water it holds—connecting real-world experience with the math behind volume.

1. **Materials Needed:**

   Calculator

   Rulers or measuring tapes

   Water and a measuring jug (optional but powerful visual!)

   Real plastic buckets, measuring cups, or flowerpots (frustum-shaped)

   Worksheets for dimensions and calculations

2. Review the volume formula of a cone and note that the frustum is a cone with the top sliced off.

   Observe real-life frustums: buckets, lampshades, party hats cut short, flower pots, juice glasses, etc.

3. Students work in pairs or small groups. They measure:

   Diameter (then radius) of top opening: **R**

   Diameter (then radius) of bottom: **r**

   Height of the container: **h**

   Measure and record the dimensions label the top radius R and bottom radius r and the height h in your worksheet and record all measurements in cm..

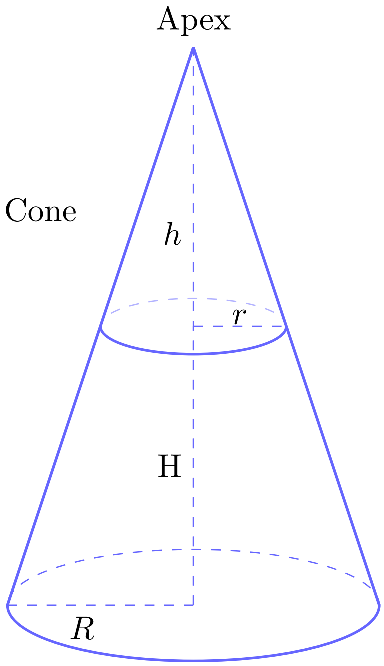
4. Fill the bucket with water and pour it into a measuring jug to find it's actual volume in liters. Then calvulate using the formula and find the volume's capacity in liters.

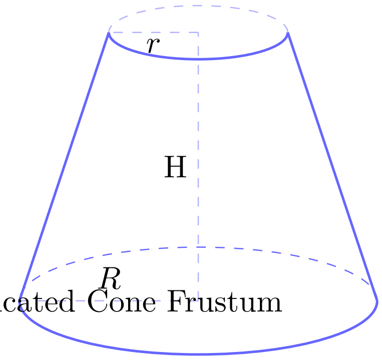
Key Takeaway

A frustum is a cone or pyramid is cut parallel to its base, removing the top portion. This results in a truncated shape with two parallel bases one smaller than the other e.g a lampshade

Types of frustums

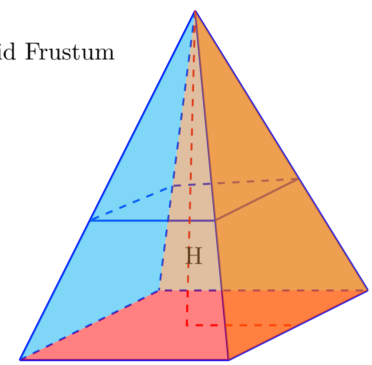
Full Cone

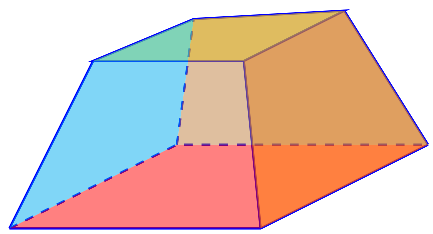




Truncated Cone Frustum

Pyramid Frustum



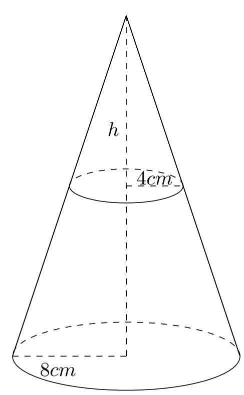


Truncated Pyramid Frustrum

Example 2.7.63 A frustum of a cone has a top radius of 4 cm, a bottom radius of 8 cm, and a height of 10 cm.

a. Find the slant height of the frustum.

b. Find the volume of the frustum.



Solution. Using the Pythagoras theorem, the slant height ℓ is given by:

$$\begin{aligned}\ell =&\sqrt{8^2+10^2}\\ =&12.80\,\text{cm}\end{aligned}$$

Slant height $\ell = 12.80\,\text{cm}$

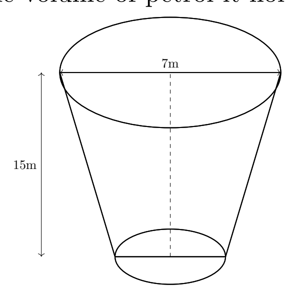
Finding the volume. $V = \frac{1}{3}\pi h(r^2 + R^2 + Rr)$

$$\begin{aligned}V =&\frac{1}{3}\times\frac{22}{7}\times 10\,\text{cm}\times(4^2+8^2+(8\times4))\,\text{cm}\\ =&\frac{22}{21}\times 10\,\text{cm}\times(16\,\text{cm}+64\,\text{cm}+32\,\text{cm})\\ =&\frac{220}{21}\times 112\,\text{cm}^2\\ =&1173.33\,\text{cm}^3\end{aligned}$$

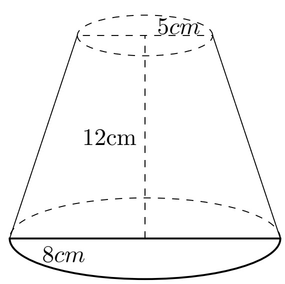
□

Exercise

1. Shell petrol station has the shape of an inverted right circular cone of the radius 7 m and height 15m dug underground in the shape of the figure shown below. Calculate the volume of petrol it holds when full in litre.



2. A frustum of a cone has a base radius of 8 cm, a top radius of 5 cm, and a height of 12 cm. Find its volume.



3. A square pyramid is cut into a frustum, where the original height was 18 cm, and the truncated top part has a height of 6 cm. The base side length is 12 cm, and the top side length is 6 cm. Find the volume of the frustum.

4. A frustum of a cone has a volume of 900 cm^3, a base radius of 10 cm, and a top radius of 6 cm. Find its height.

5. An A improvised jerrycan shaped like a frustum of a cone with a top radius of 20 cm, a bottom radius of 15 cm, and a height of 30 cm. How many liters of water can the bucket hold?

2.7.17 Volume of Composite solids

Activity 2.7.22 Constructing a real-world object using basic solids (cylinder, cube, cone, hemisphere, etc.) and calculate the total volume.

- **Materials needed:**

  Building blocks or 3D shape cut-outs (foam, paper nets, or toy blocks)

  Rulers or measuring tapes

  Worksheets for drawings and calculations

  Volume formula sheet

- Review volume formulas for Cube/cuboid, cylinder, cone, sphere, Hemisphere, triangular prisms and pyramids.

- Work in groups of atleast four members.

- Build UP a structure using 2 to 3 shapes. For example

  A lighthouse (cylinder + cone)

  An ice cream cone (cone + hemisphere)

  A mailbox (cuboid + half-cylinder)

  A robot body (cuboid + cylinder arms + sphere head)

  After building up draw the structure, label the parts and measure dimensions. Then alculate the volume of each solid part using the correct formulas.

  Then they add all volumes to get the total.

Key Takeaway

A composite solid is a three-dimensional shape made up of two or more simple solids (such as cubes, cylinders, cones, spheres, prisms, and pyramids). To find the volume of a composite solid, the volumes of the individual solids are calculated and then either added or subtracted, depending on the situation.

- Identify the simple solids that make up the composite solid

- Calculate the volume of each individual solid using the appropriate formula

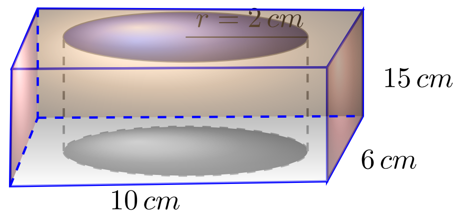
- Add or subtract the volumes i.e:

  If the solids are joined together, add their volumes.

  If a part of one solid is removed (e.g., a hole), subtract its volume from the total.

- Examples of Composite Solids include Cylinder with a Hemisphere on Top, Rectangular Prism with a Cylindrical Hole etc.

Example 2.7.64 A rectangular prism has dimensions length = 10 cm, width = 6 cm, and height = 15 cm. A cylindrical hole of radius 2 cm passes vertically through the entire height of the prism. Find the volume of the remaining solid after the hole is removed.



Solution. Step 1: Find the Volume of the Rectangular Prism

$$\begin{aligned} V_{\text{prism}} &= \text{length} \times \text{width} \times \text{height} \\ &= 10\,\text{cm} \times 6\,\text{cm} \times 15\,\text{cm} \\ &= 900\,\text{cm}^3 \end{aligned}$$

Step 2: Find the Volume of the Cylindrical Hole

$$\begin{aligned} V_{\text{cylinder}} &= \pi r^2 h \\ &= 3.14 \times (2\,\text{cm})^2 \times 15\,\text{cm} \\ &= 188.4\,\text{cm}^2 \end{aligned}$$

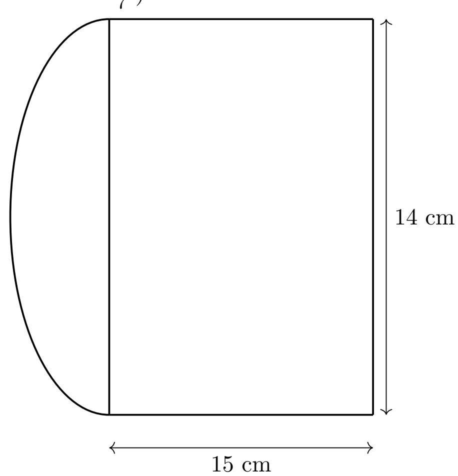
Step 3: Find the Volume of the Remaining Solid

$$\begin{aligned} V_{\text{Total volume remaining}} &= V_{\text{prism}} - V_{\text{cylinder}} \\ &= (900 - 188.4)\,\text{cm}^3 \\ &= 711.6\,\text{cm}^3 \end{aligned}$$

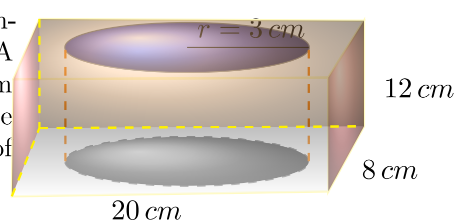
The volume of the remaining solid after the hole is removed is 711.6cm^3. □

Exercise

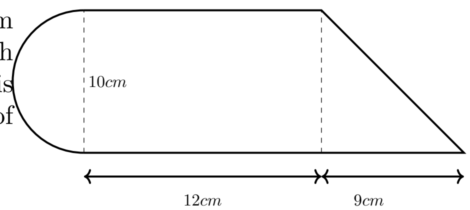
1. A cylinder has a diameter of 14 cm and a height of 15 cm. A hemisphere with the same radius is attached to the top. Find the total volume of the composite solid. (Take $\pi = \frac{22}{7}$)



2. A rectangular prism has dimensions 12 cm by 8 cm by 20 cm. A cylindrical hole with a radius of 3 cm is drilled vertically through the entire height of the prism. Find the volume of the remaining prism .



3. A cylinder has a radius of 5 cm and a height of 12 cm. A cone with the same radius and a height of 9 cm is placed on top. Find the total volume of the solid.



4. A solid sphere with a radius of 6 cm is completely enclosed in a cube. Find the volume of the space inside the cube but outside the sphere

5. A rectangular prism has a base of 10 cm by 6 cm and a height of 15 cm. A square pyramid with a base of 10 cm by 10 cm and a height of 8 cm is placed on top of the prism. Find the total volume of the solid.

Technology 2.7.65 "Geometry is not just about shapes—it's about seeing the invisible. With technology, we don't just see more clearly—we explore more deeply." *Adapted for modern mathematics education*

Visit the links below to have interactive exercises that will sharpen your mathematical skills to the uttermost!

https://www.mathsisfun.com/geometry/plane-geometry.html

Checkpoint 2.7.66 This question contains interactive elements.

Checkpoint 2.7.67 Find the volume of the figure below.const board = JXG.JSXGraph.initBoard('box', { boundingbox: [-2, 5, 5, -2], showCopyright: false, axis: false, showNavigation: false }); const side = 2.5; // Define points let A = board.create('point', [0, 0], {visible: false}); let B = board.create('point', [side, 0], {visible: false}); let C = board.create('point', [side, side], {visible: false}); let D = board.create('point', [0, side], {visible: false}); let A1 = board.create('point', [0.8, 1], {visible: false,'fixed':true}); let B1 = board.create('point', [side + 0.8, 1], {visible: false,'fixed':true}); let C1 = board.create('point', [side + 0.8, side + 1], {visible: false,'fixed':true}); let D1 = board.create('point', [0.8, side + 1], {visible: false,'fixed':true}); // Create edges board.create('polygon', [A, B, C, D], {borders: {strokeColor: 'blue','fixed':true}}); board.create('polygon', [A1, B1, C1, D1], {borders: {strokeColor: 'blue','fixed':true}}); board.create('line', [A, A1], {straightFirst: false, straightLast: false, strokeColor: 'blue','fixed':true}); board.create('line', [B, B1], {straightFirst: false, straightLast: false, strokeColor: 'blue','fixed':true}); board.create('line', [C, C1], {straightFirst: false, straightLast: false, strokeColor: 'blue','fixed':true}); board.create('line', [D, D1], {straightFirst: false, straightLast: false, strokeColor: 'blue','fixed':true}); // Label the 3 visible edges with 13 cm board.create('text', [1.2, -0.2, "55 cm"],{'fixed':true}); board.create('text', [3.5, 2, "55 cm"],{'fixed':true}); board.create('text', [3, 0.4, "55 cm"],{'fixed':true}); Volume =________cm^2

Answer. 166375

Solution. The volume of the cube is given by:

Volume = Side^3

Therefore,

$$\begin{aligned}\text{Volume} &= \text{Side}^3\\ &= 55^3\\ &= 55 \times 55 \times 55\\ &= 166375\,\text{cm}^3\end{aligned}$$

Checkpoint 2.7.68 This question contains interactive elements.

Checkpoint 2.7.69 A large cubic box has a side length of 480 cm. How many smaller cubic boxes, each with a side length of 30 cm, can fit inside the large box?

Volume of the large box =________cm^2
Volume of the small box =________cm^2
Number of small boxes =________cm

Answer 1. 110592000

Answer 2. 27000

Answer 3. 4096

Solution. Given:

- Large cube side = 480 cm
- Small cube side = 30 cm

Volume of a cube with side (S) a is $V = S^3$**Volume of the large box**

$$\begin{aligned}\text{Volume} &= 480^3\\ &= 480 \times 480 \times 480\\ &= 110592000\,\text{cm}^3\\ \text{Therefore,}&\\ \text{Volume of large box} &= 110592000\,\text{cm}^3\end{aligned}$$

Volume of the small box

$$\begin{aligned}\text{Volume} &= 30^3\\ &= 30 \times 30 \times 30\\ &= 27000\,\text{cm}^3\\ \text{Therefore,}&\\ \text{Volume of small box} &= 27000\,\text{cm}^3\end{aligned}$$

How many small boxes fit inside the large box Divide the big volume by the small volume:

$$\begin{aligned}110592000 \div 27000 &= 4096\\ &\approx 4096\\ \text{Therefore,}&\\ \text{Number of small boxes} &= 4096\end{aligned}$$

2.8 Vectors I

In mathematics, ***vectors*** are a fundamental concept that goes beyond numbers. Suppose I ask, "How far is your home from school?" One possible response is " 2 kilometers." However, I can't get to your home with just this information. I would also need to know the direction, whether it is east, west, northeast, or south. This combination of both distance and direction is what a vector represents.

Pilots use vectors to calculate the distance and direction they need to take inorder to travel from one location to another. In this section, we will explore how to use vectors and apply different operations on them.

2.8.1 Vector and Scalar Quantities

In our daily lives, we measure many things using just a single number, such as the temperature of a room or the mass of an object. However, for some physical quantities, knowing "how much" is not enough; we also need to know "which way". For example, knowing a hospital is 5 km away is not helpful unless you also know the direction to travel. This section explores the key differences between quantities that only have ***magnitude*** and those that have both ***magnitude*** and ***direction***.

Activity 2.8.1 ***Work in groups***

(a) Imagine it is a normal school day and students are playing football during games time. Suddenly, one student gets injured and needs to be taken to the nearest hospital immediately. A boda boda rider has agreed to help, but he does not know the way.

(b) Your task is to draw a clear and simple map that will guide the boda boda rider from the school to the nearest hospital. Begin by imagining that you are standing at the school gate and think carefully about the route you would take to reach the hospital. As you draw the road, show all the turns you would make, whether left, right, or straight ahead.

(c) Include at least three landmarks that the rider would see along the way, such as a market, a police station, a church, or a large tree. These landmarks should be placed in their correct positions to help the rider know that he is on the right path. Use arrows to show the direction the boda boda should follow and label any roads if you know their names.

(d) Finally, look at your map and ask yourself whether someone who has never been in the area could use it to reach the hospital without getting lost. If necessary, add a compass showing North, South, East, and West to make your directions clearer.

Definition 2.8.1 ***A vector*** is a quantity that has ***both*** magnitude and direction, whereas ***a scalar*** is a quantity which has ***only*** magnitude.

Examples of ***vector*** quantities include force, velocity, and displacement, while ***scalar*** quantities include mass,temperature, and speed. ◊

Key Takeaway 2.8.2

Table 2.8.3 Difference Between Vector and Scalar Quantities

Feature	**Vector Quantity**	**Scalar Quantity**
Definition	Has both magnitude and direction	Has only magnitude
Examples	Force,Acceleration,Displacement	Distance,Temperature,Mass

Checkpoint 2.8.4 Scalar or Vector Quantity. Load the question by clicking the button below.Identify whether each of the following physical quantities is a scalar or a vector.

1. The capacity (volume) of a water tank. [Scalar | Vector]
2. The force exerted by a person pushing a heavy box. [Scalar | Vector]
3. The height of a classroom flagpole in meters. [Scalar | Vector]
4. The relative velocity of two aircraft. [Scalar | Vector]

Answer 1. Scalar

Answer 2. Vector

Answer 3. Scalar

Answer 4. Vector

Solution.

1. Volume measures the amount of space an object occupies without any direction, making it a scalar.
2. Force has both a size (magnitude) and a specific direction in which it is applied, making it a vector.
3. Height is a measurement of length. Length only has a size (magnitude) but no specific direction, so it is a scalar.
4. Velocity always implies both a speed and a specific direction of movement, so it is a vector quantity.

2.8.2 Representation of Vectors and Vector Notation

Since ***vectors*** describe both ***magnitude*** and ***direction***, we cannot represent them using simple numbers alone. To communicate mathematical ideas clearly, we need a standardized way to draw and write vectors so that everyone interprets them in exactly the same way. The following activity will help you explore how to visualize and record these movements before we define the formal notation.

Activity 2.8.2 ***Work in groups***

(a) Go into the school compound field, choose a starting point and mark it as point A.

(b) Standing at point A, locate the ***north***, ***south***, ***west***, and ***east*** directions.

(c) From point A, walk 20 steps to the ***north*** direction and mark it as point B.

(d) From point B, walk 15 steps to the ***east*** direction and mark it as point C.

(e) On a piece of graph paper, draw a sketch that shows your path from point A to B and then from B to C.

(f) Think of a way on how you can represent the movement from point A to B using notation, also from point B to C.

(g) Suppose you now move in a reverse way from point C to point A following the same path. How would you represent that movement using notation?

(h) Discuss your ideas and representations with the rest of the learners in class.

Key Takeaway 2.8.5 We can represent a vector by a line drawn between two points as shown in Figure 2.8.6 below:

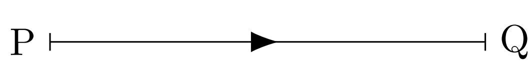


Figure 2.8.6

Vector notation is a way of representing quanties that have both magnitude and direction. Vector **PQ** can be denoted as $\underset{\sim}{PQ}$ or $\overrightarrow{PQ}$ or **PQ**.The magnitude of vector **PQ** is represented as $|\mathbf{PQ}|$. In this case, we refer to P as ***initial point*** and Q as the ***terminal point***.

Additionally, a vector can also be represented using a single small letter,such as **a** or $\underset{\sim}{\mathbf{a}}$. In Figure 2.8.7 below, we can represent a vector from a point P to point Q as $\overrightarrow{PQ} = \mathbf{a} = \underset{\sim}{\mathbf{a}}$

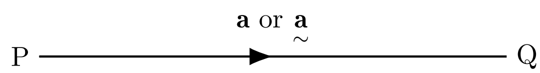


Figure 2.8.7

Similarly, if the direction of the vector is reversed, from point Q to point P the vector is represented as $\overrightarrow{QP} = -\mathbf{a} = -\underset{\sim}{\mathbf{a}}$

Exercises

1. A student is participating in a treasure hunt[1]on the school sports field. They start at the center of the field, which we will call Point **M**. The student walks 30 meters directly North to a flag at Point **N**.

i) Write the vector representing this movement using arrow notation.

ii) If this vector is assigned the letter **b**, write it using tilde notation.

Answer.

2.8.3 Equivalent Vectors

Now that you understand how to represent vectors using magnitude and direction, we can explore how to compare them. Is it possible for two vectors to be considered "*equal*" even if they are located in different places on a graph? In this section, you will examine the specific conditions regarding ***length*** and ***direction*** that must be met for ***two*** vectors to be called ***equivalent***.

Activity 2.8.3 ***Work in groups***
What you require: Graph paper,ruler

(a) Draw the x and y axis on the graph paper.

(b) Plot the points $A(0, 4), B(3, 4), C(0, 2)$ and $D(3, 2)$.

(c) Draw a line to connect point A and B, add an arrow pointing to point B.

(d) Draw a line to connect point C and D, add an arrow pointing to point D.

(e) Look at the two arrows you have drawn. Do they look like "clones" (exact copies) of each other?

(f) Imagine sliding the vector **AB** straight down without turning it.

- If you move point A so it sits exactly on top of point C, where does point B land?

[1]**A treasure hunt** is a game or activity where participants follow a series of clues, riddles, or maps to locate a hidden prize or "treasure".

- Does it land exactly on point D?

(g) Calculate the length of both vectors and compare the direction they are pointing. What two properties do vector **AB** and vector **CD** have in common?

(h) Since these vectors share the exact same properties, discuss with your group what relationship exists between them.

Key Takeaway 2.8.8 ***Two*** or ***more*** vectors are said to be equivalent if they satisfy the following conditions:

(a) They have same ***magnitude***.

(b) They point in the ***same direction***.

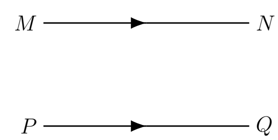


Figure 2.8.9

In Figure 2.8.9 above, **MN** = **PQ** since they have same direction and equal magnitude.

Example 2.8.10 Using Figure 2.8.11 below, determine whether vector **AB** and **DC** are equivalent.

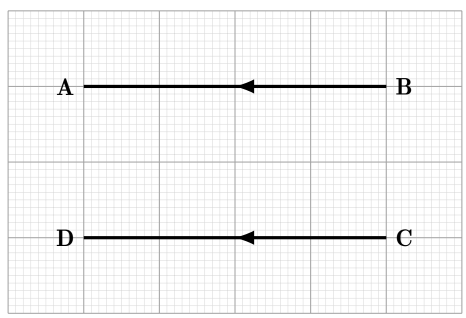


Figure 2.8.11

Solution. Vector **AB** and **DC** are equivalent because they have the same magnitude, $|\mathbf{AB}| = |\mathbf{DC}|$, and they point in the same direction □

Exercises

1. Is it possible for two vectors to have the same direction but ***not*** to be equivalent? Explain your answer.

2. In Figure 2.8.12, identify pairs of ***equivalent*** vectors and ***non-equivalent*** vectors.

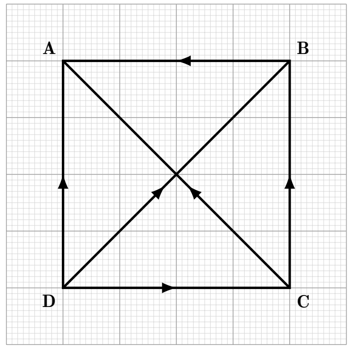


Figure 2.8.12

3. Draw two vectors that have the same magnitude and direction but start at different points.

Checkpoint 2.8.13 This question contains interactive elements.

Checkpoint 2.8.14 This question contains interactive elements.

2.8.4 Addition of Vectors

Activity 2.8.4 ***Work in groups***
What you require: Graph paper,ruler

(a) Draw the x and y axis on the graph paper.

(b) Draw vector **AB** from point $A(0,0)$ to point $B(2,2)$.

(c) Draw vector **BC** from point $B(2,2)$ to point $C(5,2)$.

(d) Count the number of units moved horizontally (along the x axis) from the starting point A to the final point C.

(e) Similarly, count the number of units moved vertically (along the y axis) from point A to point C.

(f) Write the resultant displacement in coordinate form $\begin{pmatrix} x \\ y \end{pmatrix}$, where x represents displacement along the x axis and y represents displacement along the y axis.

(g) Discuss and share your findings with the rest of the class.

Key Takeaway

Consider a displacement from point P to point Q, followed by another displacement from point Q to point N. The total resultant displacement from P to N is obtained by adding the two vectors sequentially.

This can be expressed as:

$$\mathbf{PN} = \mathbf{PQ} + \mathbf{QN}$$

where:

1. $\mathbf{PQ} = \mathbf{r}$ represents the first displacement.
2. $\mathbf{QN} = \mathbf{s}$ represents the second displacement.

Therefore, $\mathbf{PN} = \mathbf{r} + \mathbf{s}$

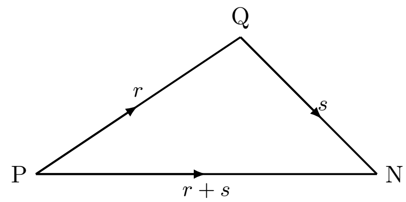


Figure 2.8.15

We can also re-arrange the two vectors and add them together as shown in Figure 2.8.16 below.

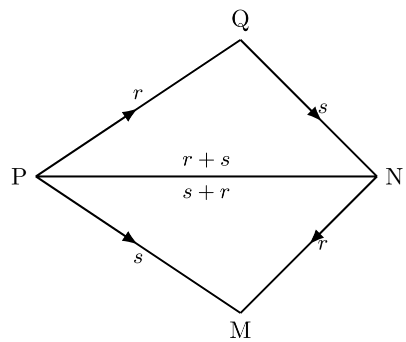


Figure 2.8.16

Example 2.8.17 In Figure 2.8.18 below, find vector **AD** in terms of **a** and **b**.

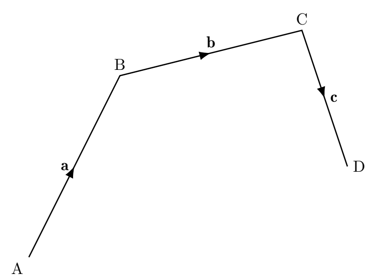


Figure 2.8.18

Solution.

$$\begin{aligned}\overrightarrow{AD} &= \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} \\ &= \mathbf{a} + \mathbf{b} + \mathbf{c}\end{aligned}$$

Thus, $\overrightarrow{AD} = \mathbf{a} + \mathbf{b} + \mathbf{c}$ □

Exercises

1. $PQNM$ is a square with vectors **PQ** and **PM** given as **a** and **b** respectively, as shown in Figure 2.8.19 . Express the **PN** and **MQ** vectors in terms of **a** and **b**

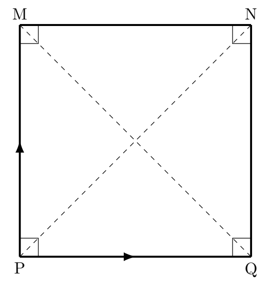


Figure 2.8.19

2. Use Figure 2.8.20 to answer the questions below:

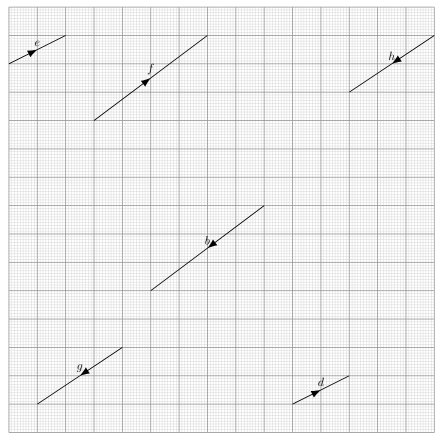


Figure 2.8.20

(a) List pairs of equal vectors.

(b) Name pairs of vectors with equal magnitude but opposite directions.

(c) illustrate the sums of the following vectors graphically:

(i) $\mathbf{e} + \mathbf{d}$

(ii) $\mathbf{f} + \mathbf{g}$

(iii) $\mathbf{b} + \mathbf{d}$

(iv) $\mathbf{g} + \mathbf{h}$

3. Given the vectors $\mathbf{a} = \binom{2}{3}$ and $\mathbf{b} = \binom{4}{-1}$, find $\mathbf{a} + \mathbf{b}$ and illustrate the solution graphically.

Checkpoint 2.8.21 Exercise

In the rectangle $ABCD$ the side AB is represented by the vector $\underline{p}$ and the side BC is represented by $\underline{q}$. Find an expression in terms of $\underline{p}$ and $\underline{q}$ for:

1. The diagonal $\overrightarrow{AC}$: _______

2. The diagonal $\overrightarrow{BD}$: _______

Note: you should enter your answer in terms of the letters p and q without underline or boldface.

Answer 1. $q + p$

Answer 2. $q - p$

Solution.

1. The diagonal $\overrightarrow{AC}$ is the sum of the sides $\overrightarrow{AB}$ and $\overrightarrow{BC}$. Therefore

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \underline{p} + \underline{q}$$

2. The diagonal $\overrightarrow{BD}$ is the sum of the sides $\overrightarrow{BC}$ and $\overrightarrow{CD}$. Therefore

$$\overrightarrow{BD} = \overrightarrow{BC} + \overrightarrow{CD} = \underline{q} - \underline{p}$$

Checkpoint 2.8.22 Exercise

In triangle OAB the point P divides AB internally in the ratio $m : n$.

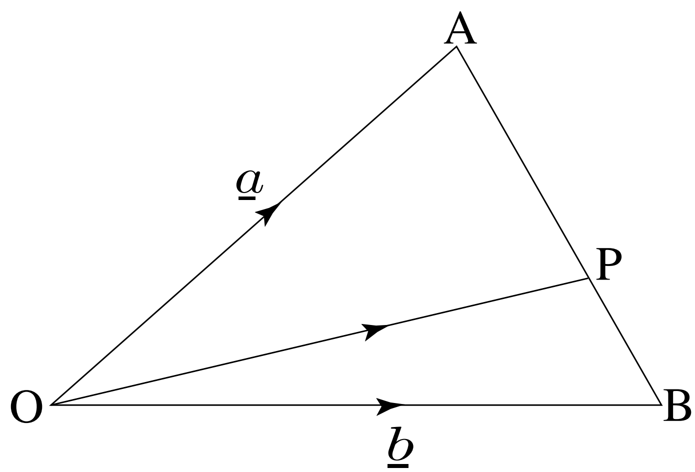
If $\overrightarrow{OA} = \underline{a}$ and $\overrightarrow{OB} = \underline{b}$, then find an expression for $\overrightarrow{OP}$ in terms of $\underline{a}$ and $\underline{b}$.

You might find it helpful to draw a diagram showing this triangle.

$\overrightarrow{OP} =$ _______________

Answer. $\frac{(b-a)m}{n+m} + a$

Solution. The figure below shows the triangle with the vector $\overrightarrow{OP}$ marked with a dotted line within.



From the diagram we can see $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$. We are told $\overrightarrow{OA} = \underline{a}$ and so we need to find $\overrightarrow{AP}$. We know that P divides $\overrightarrow{AB}$ in the ratio $m : n$ so

$$\overrightarrow{AP} = \frac{m}{m+n}\left(\overrightarrow{AB}\right),$$

as

$$\overrightarrow{AB} = \underline{b} - \underline{a}$$

we find

$$\overrightarrow{AP} = \frac{m}{m+n}\left(\underline{b} - \underline{a}\right)$$

and therefore,

$$\overrightarrow{OP} = \underline{a} + \frac{m}{m+n}\left(\underline{b} - \underline{a}\right).$$

Note: You can arrive at an equivalent expression by starting from $\overrightarrow{OP} = \overrightarrow{OB} + \overrightarrow{BP}$.

Checkpoint 2.8.23 Exercise

Given a regular hexagon A, B, C, D, E, F, with O at the centre.

Let $\underline{a} = \vec{OA}$ and $\underline{b} = \vec{OB}$.

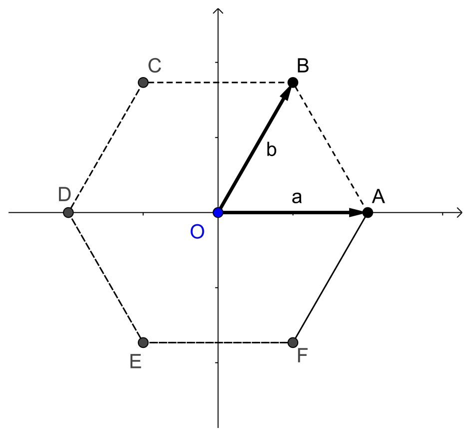


Figure 2.8.24

Express each of the following in terms of $\underline{a}$ and $\underline{b}$.

1. $\vec{BC} =$ _____
2. $\vec{BC} + \vec{DE} + \vec{FA} =$ _____
3. $\vec{CF} =$ _____

Write $\underline{a}$ and $\underline{b}$ as `a` and `b`, respectively.

Answer 1. $-a$

Answer 2. 0

Answer 3. $2\,(a-b)$

Solution.

1. From the properties of the regular hexagon we know that $\vec{CB}$ is parallel to $\underline{a}$ and they have the same length. So, $\vec{BC} = -\underline{a}$.
2. We notice that $\vec{CO}$ is a translation of $\vec{DE}$ and $\vec{OB}$ is a translation of $\vec{FA}$. So, $\vec{BC} + \vec{DE} + \vec{FA} = \vec{BC} + \vec{CO} + \vec{OB} = 0$.
3. From the properties of the hexagon we know that the diagonal $\vec{CF}$ is parallel to the side $\vec{BA}$ and has twice it's length. So $\vec{CF} = 2\vec{BA} = 2(\underline{a} - \underline{b})$.

Checkpoint 2.8.25 Exercise

Given that $\underline{a} = 11\underline{i} - 5\underline{j}$ and $\underline{b} = 13\underline{i} + 6\underline{j}$ find $\underline{a} + \underline{b}$.

$\underline{a} + \underline{b} =$ _____ $\underline{i}$ + _____ $\underline{j}$

Answer 1. 24

Answer 2. 1

Solution. Given that $\underline{a} = 11\underline{i} - 5\underline{j}$ and $\underline{b} = 13\underline{i} + 6\underline{j}$

$$\begin{aligned} \underline{a} + \underline{b} &= (11 + (13))\,\underline{i} + (-5 + (6))\,\underline{j}, \\ &= 24\underline{i} + \underline{j}. \end{aligned}$$

2.8.5 Multiplication of Vectors by a Scalar

Activity 2.8.5 ***Work in groups***

(a) On a graph paper, draw the x axis and y axis.

(b) Draw a directed line passing through point $A(0,2)$ and $B(2,2)$.

(c) From point $B(2,2)$, draw directed line to point $C(4,2)$.

(d) Determine the coordinate representations of $\overrightarrow{AB}$ and $\overrightarrow{AC}$..

(e) How does $\overrightarrow{AB}$ relate to $\overrightarrow{AC}$?

(f) Discuss and share your findings with the rest of the class.

Key Takeaway

Positive Scalar

In Figure 2.8.26, the vector **PQ** is represented as **a**. When we multiply **a** by a positive scalar, say 2, the length of the vector doubles, making it **2a** as shown in Figure 2.8.27. The direction of the vector remains unchanged, but its magnitude increases.

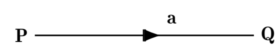


Figure 2.8.26

In Figure 2.8.27, the vector **PN** is given by: $\mathbf{PN} = \mathbf{a} + \mathbf{a} = 2\mathbf{a}$. This means **PN** has the same direction as **PQ**, but its magnitude twice that of **PQ**.

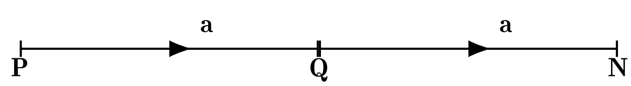


Figure 2.8.27

Negative Scalar

Consider the vector **AB**,denoted as **a**, in Figure 2.8.28 The vector points to the right and has a magnitude of **a**.

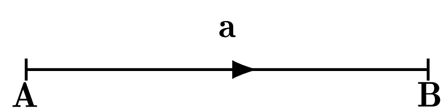


Figure 2.8.28

In Figure 2.8.29, the vector **AB** is obtained by multiplying **AB** by -2, giving:

$$\mathbf{AC} = -\mathbf{2} \times \mathbf{a} = -\mathbf{2a}$$

This means that **AC** has ***twice*** the magnitude of **AB**, but its direction is ***reversed***.

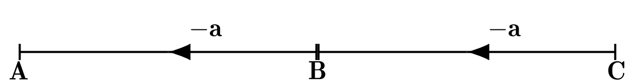


Figure 2.8.29

Multiplying a vector by a ***negative scalar*** reverses its direction, making it point in the ***opposite direction***.

Zero Scalar

When a vector **a**, as shown in Figure 2.8.30, is multiplied by 0, its magnitude becomes 0, resulting in a ***zero vector***.

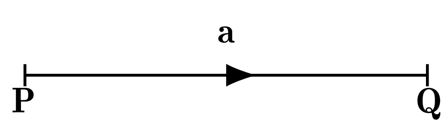


Figure 2.8.30

$$\mathbf{a} \cdot 0 = 0$$

Example 2.8.31 Given the vectors $u = 2p+5q$ and $v = p-3q$, express $3u+2v$ in terms of p and q :

Solution. Substitute the given expressions for u and v.

$$\begin{aligned} 3u + 2v &= 3(2p + 5q) + 2(p - 3q) \\ &= 6p + 15q + 2p - 6q \end{aligned}$$

Combine like terms (terms with p and terms with q).

$$\begin{aligned} 3u + 2v &= (6p + 2p) + (15q - 6q) \\ &= 8p + 9q \end{aligned}$$

□

Exercises

1. Simplify the following:

(a) $5x + 3y - z + 2(3x - z) + (8x - 6y)$

(b) $(\mathbf{a} - \mathbf{b}) + (c - \mathbf{a}) + (\mathbf{b} - c)$

(c) $4m - 2n + 5(k - m) + 2(m + n)$

2. Given that $x = 3m - n$ and $y = n + 4m$, express the following vectors in terms of m and n :

(a) $3x$

(b) $\frac{2}{3}y$

(c) $6x - 9y$

(d) $3y - x$

3. A pentagon $ABCDE$ with $\overrightarrow{AB} = m$, $\overrightarrow{BC} = n$, and $\overrightarrow{CD} = k$. Express the following vectors in terms of m, n, and k :

(a) $\overrightarrow{AC}$

(b) $\overrightarrow{AD}$

(c) $\overrightarrow{AE}$

Checkpoint 2.8.32 Exercise

Simplify the following:

1. $6\underline{a} + 2\underline{a} =$_____
2. $6(4\underline{b}) =$_____
3. $4(\underline{q} + \underline{r}) =$_____

Write $\underline{a}$ and $\underline{b}$ as `a` and `b`, respectively.

Answer 1. $8a$

Answer 2. $24b$

Answer 3. $4r + 4q$

Solution.

1. The second rule $6\underline{a} + 2\underline{a}$ can be simplified to $8\underline{a}$.
2. Using the third rule, $6(4\underline{b}) = 24\underline{b}$.
3. Using the first rule, $4(\underline{q} + \underline{r}) = 4\underline{q} + 4\underline{r}$.

2.8.6 Column Vectors

Activity 2.8.6 ***Work in groups***

(a) Draw the x axis and y axis on the graph paper.

(b) Plot point $A(1,1)$ and point $B(5,3)$ on the graph.

(c) Draw a directed line from point A to point B.

(d) Represent vector AB in terms of its components as $\begin{pmatrix} x \\ y \end{pmatrix}$ where x is the horizontal displacement and y is the vertical displacement.

(e) Discuss and share your findings with the rest of the class.

Key Takeaway

A ***vector*** expressed in the form of $\begin{pmatrix} a \\ b \end{pmatrix}$, where a is the horizontal displacement along the x axis and **b** is the vertical displacement along the y axis is known as a ***column vector.***

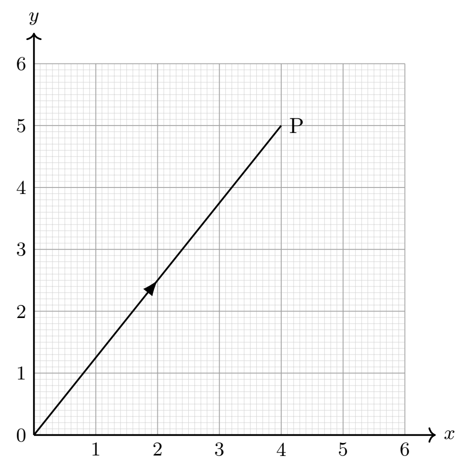


Figure 2.8.33

The vector **OP** illustrates a displacement from the origin $O(0,0)$ to the point $P(4,5)$. This consist of a horizontal displacement of 4 units along the x axis and a vertical displacement of 5 units in the y axis.

Example 2.8.34 Given that: $\mathbf{a} = \binom{1}{4}$ and $\mathbf{b} = \binom{5}{3}$. Find $\mathbf{a} + \mathbf{b}$ and illustrate the solution graphically.

Solution. To determine $\mathbf{a} + \mathbf{b}$, we calculate the total displacement in both the x and y directions:

Horizontal displacement is $1 + 5 = 6$.

Vertical displacement is $4 + 3 = 7$.

Graphical Representation

Begin at the point $(1,0)$ on the grid, move 1 unit horizontally to the right and move 4 units vertically upwards and mark it as end point. Draw a directed line connecting the two points as shown in the Figure 2.8.35.

From the point $(4,0)$ on the grid, move 5 units horizontally to the right parallel to the x axis, and move 3 units vertically up and mark it as end point. Draw another directed line to join the two points.

Now, to find the resultant vector $\mathbf{a} + \mathbf{b}$, join the initial point $(1,0)$ with the final point $(7,7)$ and count the total displacements in the x and y directions.

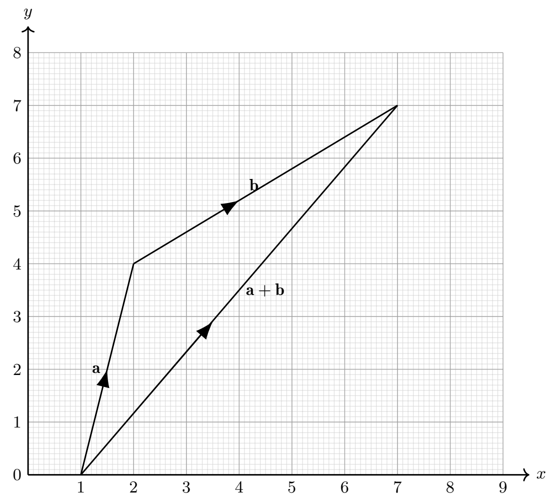


Figure 2.8.35

Therefore, $\mathbf{a}+\mathbf{b} = \binom{1}{4} + \binom{5}{3}$
$= \binom{6}{7}$ ☐

Example 2.8.36 If $\mathbf{a} = \binom{4}{7}$ and $\mathbf{b} = \binom{3}{5}$, find $2\mathbf{a}+5\mathbf{b}$.

Solution. To determine $2\mathbf{a}+5\mathbf{b}$, we multiply vector **a** by 2 and vector **b** by 5 and finally we add the resulting vectors.

$$2\mathbf{a} = 2\begin{pmatrix}4\\7\end{pmatrix} = \begin{pmatrix}2\times 4\\2\times 7\end{pmatrix} = \begin{pmatrix}8\\14\end{pmatrix}$$

$$5\mathbf{b} = 5\begin{pmatrix}3\\5\end{pmatrix} = \begin{pmatrix}5\times 3\\5\times 7\end{pmatrix} = \begin{pmatrix}15\\25\end{pmatrix}$$

Therefore, $2\mathbf{a}+5\mathbf{b} = \binom{8}{14} + \binom{15}{25} = \binom{8+15}{14+25} = \binom{23}{39}$ ☐

Exercises

1. If $a = \begin{pmatrix}2\\5\end{pmatrix}$, $b = \begin{pmatrix}4\\-1\end{pmatrix}$ and $c = \begin{pmatrix}-7\\3\end{pmatrix}$, find:

(i) $5a+7c$

(ii) $-3c+4b$

(iii) $\frac{2}{5}a-2b$

(iv) $7c-2a$

(v) $\frac{1}{4}a+\frac{3}{5}b-\frac{1}{2}c$

(vi) $5a+3c-b$

2. Given the column vectors $\mathbf{a} = \binom{3}{-2}$ and $\mathbf{b} = \binom{-1}{4}$, find the following:

(a) $\mathbf{a}+\mathbf{b}$

(b) $2\mathbf{a}-3\mathbf{b}$

Checkpoint 2.8.37 Exercise

Given $\underset{\sim}{p} = 2\underset{\sim}{i}+3\underset{\sim}{j}$ and $\underset{\sim}{q} = 3\underset{\sim}{i}+10\underset{\sim}{j}$, express $\underset{\sim}{p}$ and $\underset{\sim}{q}$ as column vectors.$\underset{\sim}{p} =$ _____ $\underset{\sim}{q} =$ _____

Answer 1. $\begin{bmatrix}2\\3\end{bmatrix}$

Answer 2. $\begin{bmatrix} 3 \\ 10 \end{bmatrix}$

Solution. In column vector notation we arrange the elements with the x-component at the top and the y-component below.

So,

$$\underline{p} = 2\underline{i} + 3\underline{j} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\underline{q} = 3\underline{i} + 10\underline{j} = \begin{bmatrix} 3 \\ 10 \end{bmatrix}$$

Checkpoint 2.8.38 Given the column vectors $\mathbf{a} = \begin{bmatrix} -5 \\ -3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$, find the following:

1. $\mathbf{a} + \mathbf{b} =$ ______
2. $2\mathbf{a} - 2\mathbf{b} =$ ______

Answer 1. $\begin{bmatrix} -10 \\ -4 \end{bmatrix}$

Answer 2. $\begin{bmatrix} 0 \\ -4 \end{bmatrix}$

Solution. Worked Solution ***Part (a)***

To add column vectors, we add the corresponding components (top with top, bottom with bottom):

$$\mathbf{a} + \mathbf{b} = \begin{bmatrix} -5 \\ -3 \end{bmatrix} + \begin{bmatrix} -5 \\ -1 \end{bmatrix} = \begin{pmatrix} -5 + (-5) \\ -3 + (-1) \end{pmatrix} = \begin{bmatrix} -10 \\ -4 \end{bmatrix}$$

Part (b)

First, we multiply each vector by its scalar:

$$2\mathbf{a} = 2\begin{bmatrix} -5 \\ -3 \end{bmatrix} = \begin{pmatrix} 2 \times -5 \\ 2 \times -3 \end{pmatrix} = \begin{bmatrix} -10 \\ -6 \end{bmatrix}$$

$$2\mathbf{b} = 2\begin{bmatrix} -5 \\ -1 \end{bmatrix} = \begin{pmatrix} 2 \times -5 \\ 2 \times -1 \end{pmatrix} = \begin{bmatrix} -10 \\ -2 \end{bmatrix}$$

Then we subtract the results:

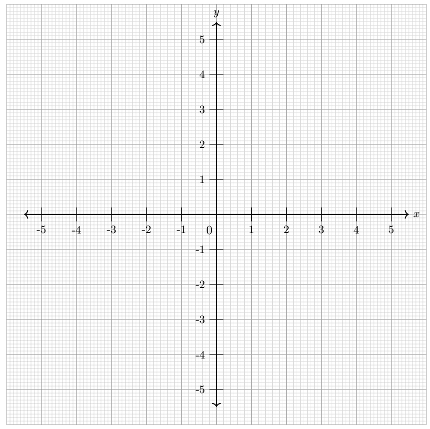
$$2\mathbf{a} - 2\mathbf{b} = \begin{bmatrix} -10 \\ -6 \end{bmatrix} - \begin{bmatrix} -10 \\ -2 \end{bmatrix} = \begin{pmatrix} -10 - (-10) \\ -6 - (-2) \end{pmatrix} = \begin{bmatrix} 0 \\ -4 \end{bmatrix}$$

2.8.7 Position Vectors

Activity 2.8.7 ***Work in groups***

What you require: Graph paper

(a) Draw the x and y axis on the graph paper as shown below.



(b) Plot the following points $A(1,1), B(3,5), C(2,1), D(4,-3)$ on the graph.

(c) Draw a directed line from A to B to represent $\overrightarrow{AB}$.

(d) Draw another directed line from D to C to represent $\overrightarrow{CD}$.

(e) Determine the position vector of B relative to point A.

(f) Determine the position vector of C relative to point D.

(g) Discuss and share your findings with the rest of the class.

Key Takeaway

In Figure 2.8.39 below, points $A(2,3)$ and $B(5,1)$ are located in the plane relative to origin point O in the plane.

The position vector of A is $\mathbf{OA} = \begin{pmatrix} 2-0 \\ 3-0 \end{pmatrix} = \binom{2}{3}$

The position vector of B is $\mathbf{OB} = \begin{pmatrix} 5-0 \\ 1-0 \end{pmatrix} = \binom{5}{1}$

Similarly, for point A in the plane its position vector $\mathbf{OA}$ is denoted by $\mathbf{a}$. Also for point B in the plane it's position vector $\mathbf{OB}$ is denoted by $\mathbf{b}$.

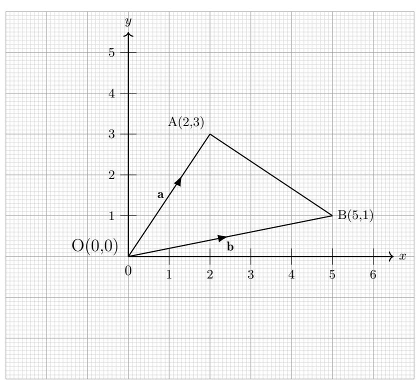
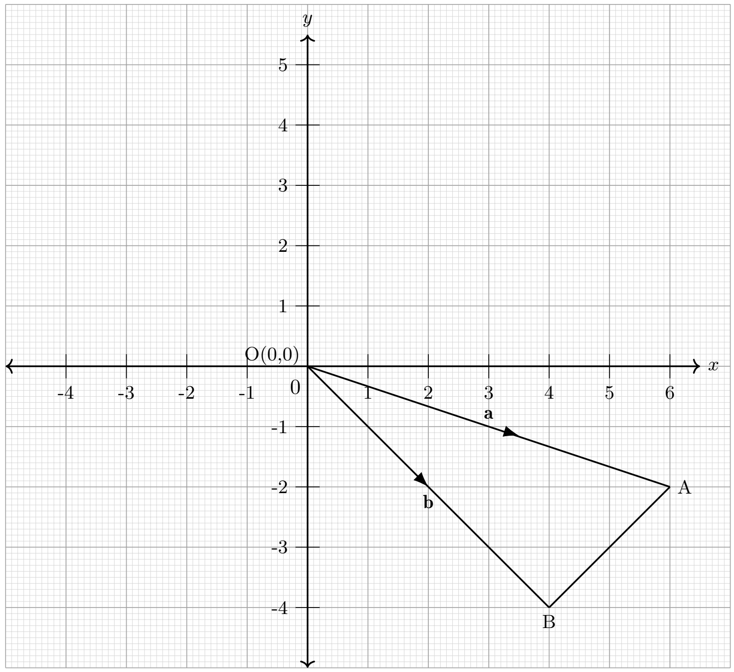


Figure 2.8.39

Example 2.8.40 Find the position vector of A and B.



Solution.

The position vector of A is **OA** $= \begin{pmatrix} 6-0 \\ -2-0 \end{pmatrix} = \begin{pmatrix} 6 \\ -2 \end{pmatrix}$

Similarly the position vector of B is **OB** $= \begin{pmatrix} 4-0 \\ -4-0 \end{pmatrix} = \begin{pmatrix} 4 \\ -4 \end{pmatrix}$ □

Exercises

1. Draw the following position vector on a graph paper:

$a = \begin{pmatrix} 10 \\ -2 \end{pmatrix}$ $b = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$ $c = \begin{pmatrix} -3 \\ 8 \end{pmatrix}$ $d = \begin{pmatrix} -11 \\ 6 \end{pmatrix}$

$e = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$ $f = \begin{pmatrix} -1 \\ 12 \end{pmatrix}$ $g = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$ $h = \begin{pmatrix} 0 \\ 13 \end{pmatrix}$

2. Use Figure 2.8.41 below to write the position vectors of points M, E, D, A.

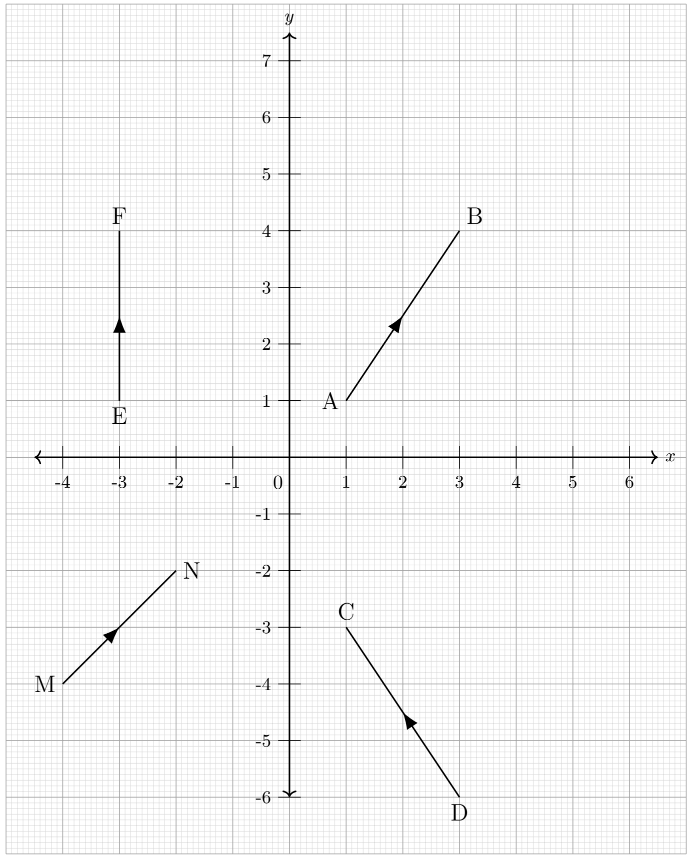


Figure 2.8.41

Checkpoint 2.8.42 Which of the following best describes a ***position vector***?

(1) A vector which may be translated to any point in space provided their direction and length remain unchanged.

(2) A vector having magnitude zero and undefined direction.

(3) A vector which always starts at the origin of the coordinate system.

(4) A vector whose length may change as long as its direction remains the same.

Answer. (3)

Solution. Worked SolutionA ***position vector*** describes the position of a point relative to the origin. Therefore, it is fixed to start at $(0, 0)$.

Other vectors (free vectors) are defined only by length and direction and can be moved anywhere in space.

2.8.8 Magnitude of a Vector

Activity 2.8.8 ***Work in groups***

What you require: Graph paper

(a) Draw the x axis and y axis on the graph paper.

(b) Mark the coordinate $(0, 0)$ as the initial point O.

(c) From Point O, move 3 units to the ***right*** along the x axis and 4 units upward in the y axis. Mark this new position as Point A.

(d) Draw a directed line from point O to point A to represent $\overrightarrow{OA}$.

(e) Use a ruler to measure the length of $\overrightarrow{OA}$.

(f) Analyze the relationship between the x displacement, y displacement, and the length of $\overrightarrow{OA}$.

(g) Discuss and share your findings with your classmates in the class.

Key Takeaway

The ***magnitude*** of $\overrightarrow{AB}$ in Figure 2.8.43 can be denoted as $|\mathbf{AB}|$. The magnitude of $\overrightarrow{AB}$ represents the distance between point A and point B.

We can represent the components of $\overrightarrow{AB}$ as $\begin{pmatrix} x \\ y \end{pmatrix}$, where x represents the horizontal displacement and y represents the vertical displacement.

We determine the magnitude of $\overrightarrow{AB}$ by applying ***Pythagorean theorem*** as shown below.

$$|\mathbf{AB}| = \sqrt{x^2 + y^2}$$

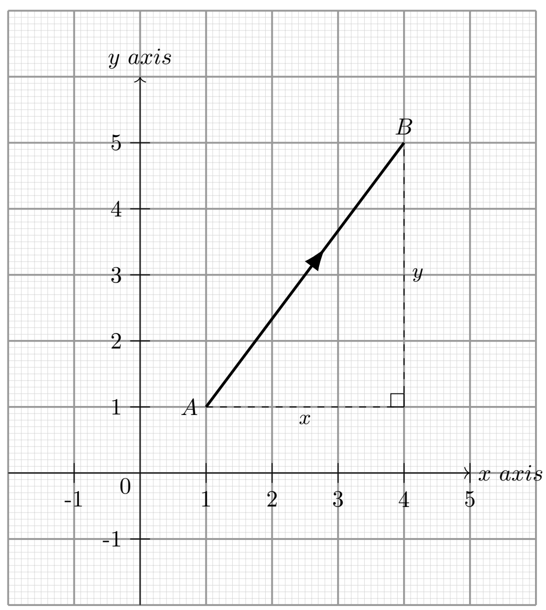


Figure 2.8.43

The magnitude of a vector is always ***positive*** since x and y components are squared, resulting in x^2 and y^2, both of which are ***non-negative***.

Example 2.8.44 Determine the magnitude of $\overrightarrow{AB}$ as shown in the Figure 2.8.45 below.

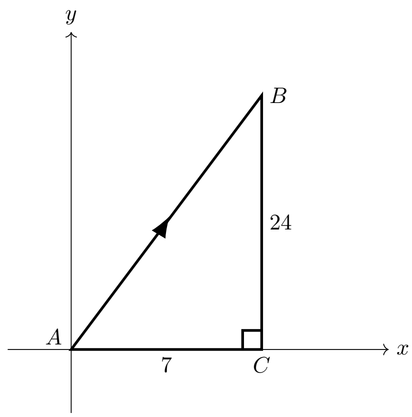


Figure 2.8.45

Solution. To find the magnitude of $\overrightarrow{AB}$, we apply pythagora's theorem to find the length **AB**.

$$\begin{aligned}|\mathbf{AB}| &= \sqrt{(\mathrm{AC})^2 + (\mathrm{CB})^2} \\ &= \sqrt{7^2 + 24^2} \\ &= \sqrt{49 + 576} \\ &= \sqrt{625} \\ &= 25\end{aligned}$$

Hence, the magnitude of $\overrightarrow{AB}$ represented as $|\mathbf{AB}|$ is 25. □

Example 2.8.46

Given that $a = \binom{2}{4}$, $b = \binom{-2}{2.5}$, $c = \binom{6}{-4}$ and $r = a + 2b - c$. Find $|r|$

Solution. $r = a + 2b - c$.

Substituting the values of a, b and c into the equation;

$r = \binom{2}{4} + 2\binom{-2}{2.5} - \binom{6}{-4}$

$r = \binom{2}{4} + \binom{-4}{5} - \binom{6}{-4}$

$r = \binom{2+(-4)-6}{4+5-(-4)}$

$r = \binom{-8}{13}$

$|r| = \sqrt{(-8)^2 + 13^2} = \sqrt{64 + 169} = \sqrt{233}$

$r = 15.26$ □

Exercises

1. Find the magnitude of each of the following vectors:

(a) $\begin{pmatrix}-6\\8\end{pmatrix}$ (b) $\begin{pmatrix}8\\15\end{pmatrix}$ (c) $\begin{pmatrix}-5\\12\end{pmatrix}$ (d) $\begin{pmatrix}3\\7\end{pmatrix}$

Checkpoint 2.8.47 Exercise

Find the modulus of each of the following vectors (writing your answer in the form of a surd, i.e. $\sqrt{x}$):

1. $\underline{r} = 13\underline{i} + 2\underline{j}$, $|\underline{r}| =$ __________

2. $\underline{s} = 12\underline{i} + 9\underline{j}$, $|\underline{s}| =$ __________

3. $\underline{t} = a\underline{i} + b\underline{j}$, $|\underline{t}| =$ __________

Answer 1. $\sqrt{173}$

Answer 2. 15

Answer 3. $\sqrt{b^2 + a^2}$

Solution. Considering the general case in part (c), for the vector, $\underline{t} = a\underline{i} + b\underline{j}$ we calculate the modulus by applying Pythagoras' theorem: $|\underline{t}| = \sqrt{a^2 + b^2}$.

Applying this we find:
For $\underline{r} = 13\underline{i} + 2\underline{j}$, $|\underline{r}| = \sqrt{173}$.
For $\underline{s} = 12\underline{i} + 9\underline{j}$ $|\underline{s}| = 15$.

Checkpoint 2.8.48 Find the magnitude of each of the following vectors. Give your answers as integers or exact surds (e.g. `sqrt(13)`).

1. $\mathbf{a} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$

   Magnitude $|\mathbf{a}| =$ ______

2. $\mathbf{b} = \begin{bmatrix} -8 \\ 0 \end{bmatrix}$

   Magnitude $|\mathbf{b}| =$ ______

Answer 1. 5

Answer 2. 8

Solution. Worked Solution The magnitude (or length) of a column vector $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ is given by the formula:

$$|\mathbf{v}| = \sqrt{x^2 + y^2}$$

Part (a)

For $\mathbf{a} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$:

$$|\mathbf{a}| = \sqrt{(4)^2 + (-3)^2}$$
$$|\mathbf{a}| = \sqrt{16 + 9}$$
$$|\mathbf{a}| = \sqrt{25} = 5$$

Part (b)

For $\mathbf{b} = \begin{bmatrix} -8 \\ 0 \end{bmatrix}$:

$$|\mathbf{b}| = \sqrt{(-8)^2 + (0)^2}$$
$$|\mathbf{b}| = \sqrt{64 + 0}$$
$$|\mathbf{b}| = \sqrt{64}$$

If the number under the square root is not a perfect square, we leave it in exact surd form: **8**

2.8.9 Midpoint of a Vector

Activity 2.8.9 ***Work in groups***

What you require: Graph paper

(a) Draw the x and y axis on the graph paper.

(b) Choose any starting point on the graph and label it as Point A. write down its coordinates.

(c) From Point A, move 6 units to the right parallel to the x axis and mark this new location as Point B. Write down its coordinates.

(d) Draw a directed line from point A to Point B to represent $\overrightarrow{AB}$.

(e) Find the midpoint of $\overrightarrow{AB}$ and label it as Point M.

(f) Identify the coordinates of Point M.

(g) Think of a way to determine coordinates of Point M without manually counting the units.

(h) Discuss and share your findings with the rest of the class.

Key Takeaway

Consider the cordinates of point P given as (x_1, y_1) and point N given as (x_2, y_2) and M is the midpoint of **PN** as shown in figure below.

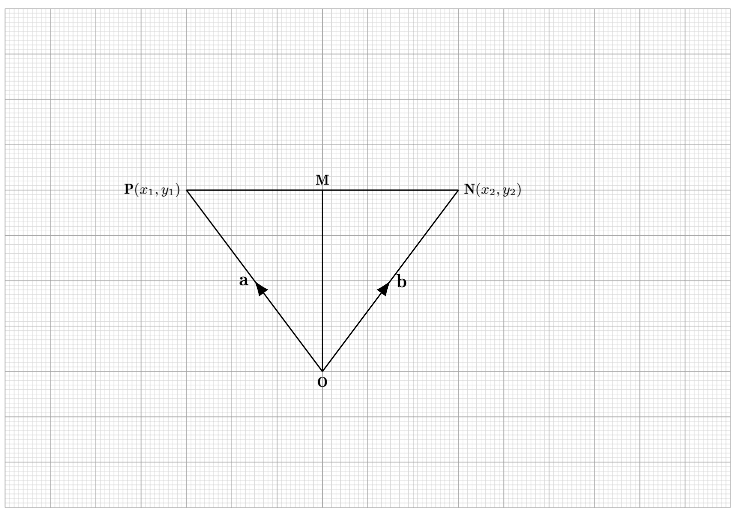


Figure 2.8.49

$$
\begin{aligned}
\mathbf{OM} &= \mathbf{OP} + \mathbf{PM} \\
&= \mathbf{a} + \frac{1}{2}\mathbf{PN} \\
&= \mathbf{a} + \frac{1}{2}(\mathbf{b} - \mathbf{a}) \\
&= \mathbf{a} + \frac{1}{2}\mathbf{b} - \frac{1}{2}\mathbf{a} \\
&= \mathbf{a} - \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b} \\
&= \frac{\mathbf{a} + \mathbf{b}}{2}
\end{aligned}
$$

But $\mathbf{a} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$

Thus, midpoint $\mathbf{M} = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

Example 2.8.50 Find the coordinates of the midpoint M of **AB** given the following points $A(6, 1)$, $B(4, 3)$.

Solution. To determine the midpoint M of $\overrightarrow{AB}$, we apply the midpoint formulae as follows.

$$\begin{aligned} \text{Midpoint } &= \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) \\ &= \left(\frac{6+4}{2}, \frac{1+3}{2}\right) \\ &= \left(\frac{10}{2}, \frac{4}{2}\right) \\ &= (5, 2) \end{aligned}$$

Thus, the coordinates of the midpoint M is $(5, 2)$ □

Exercises

1. Find the coordinates of the midpoint of **PQ** in each of the following cases:

a) $P\,(-5, 6), Q\,(3, -4)$

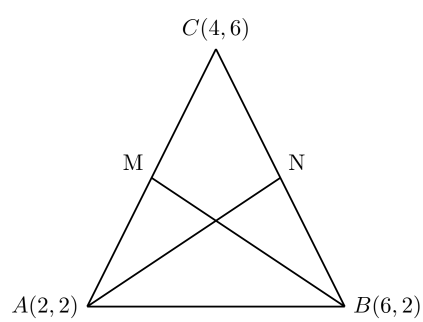
b) $P\,(1, -3), Q\,(5, 7)$

c) $P\,(-2, 4), Q\,(6, 0)$

d) $P\,(a, b), Q\,(c, d)$

2. In the figure below, ABC is a triangle where M and N are midpoints of $\overrightarrow{AC}$, $\overrightarrow{AB}$ respectively.

Triangle ABC has vertices at points $A(2, 2)$, $B(6, 2)$ and $C(4, 6)$. Determine the coordinates of M and N.



Checkpoint 2.8.51 Consider triangle PQR where $\vec{PQ} = \mathbf{u}$ and $\vec{PR} = \mathbf{v}$.

- Point S is located on PR such that $PS : SR = 2 : 4$.
- Point T lies on QR with $QT : TR = 1 : 1$.
- Point U is on PQ such that $PU : UQ = 2 : 4$.

Determine the following vectors in terms of **u** and **v**.

*(Enter **u** as* u *and **v** as* v*).*

1. $\vec{PS} =$ ______
2. $\vec{PT} =$ ______

3. $\vec{PU} =$ ______

4. $\vec{ST} =$ ______

5. $\vec{TU} =$ ______

Answer 1. $\frac{v}{3}$
Answer 2. $\frac{v-u}{2} + u$
Answer 3. $\frac{u}{3}$
Answer 4. $\frac{v-u}{2} - \frac{v}{3} + u$
Answer 5. $-\frac{v-u}{2} - \frac{2u}{3}$

Solution. Worked Solution ***(a) Finding*** $\vec{PS}$

We are given that $PS : SR = 2 : 4$. This means the line PR is divided into $2 + 4 = 6$ parts.

PS represents 2 of those parts.

$$\vec{PS} = \frac{2}{6}\vec{PR} = \frac{\mathbf{v}}{\mathbf{3}}$$

(c) Finding $\vec{PU}$

Similarly, $PU : UQ = 2 : 4$. The line PQ is divided into $2 + 4 = 6$ parts.

$$\vec{PU} = \frac{2}{6}\vec{PQ} = \frac{\mathbf{u}}{\mathbf{3}}$$

(b) Finding $\vec{PT}$

To find $\vec{PT}$, we first need to travel from P to Q, and then from Q to T: $\vec{PT} = \vec{PQ} + \vec{QT}$.

First, find $\vec{QR}$:

$$\vec{QR} = \vec{PR} - \vec{PQ} = \mathbf{v} - \mathbf{u}$$

We know $QT : TR = 1 : 1$, so QT is $\frac{1}{2}$ of QR.

$$\vec{QT} = \frac{1}{2}(\mathbf{v} - \mathbf{u})$$

Now, add them together:

$$\vec{PT} = \mathbf{u} + \frac{1}{2}(\mathbf{v} - \mathbf{u})$$

$$\vec{PT} = \frac{\mathbf{v}}{\mathbf{2}} + \frac{\mathbf{u}}{\mathbf{2}}$$

(d) Finding $\vec{ST}$

Using the triangle rule:

$$\vec{ST} = \vec{PT} - \vec{PS}$$

$$\vec{ST} = \left(\frac{v}{2} + \frac{u}{2}\right) - \left(\frac{v}{3}\right)$$

$$\vec{ST} = \frac{\mathbf{v}}{\mathbf{6}} + \frac{\mathbf{u}}{\mathbf{2}}$$

(e) Finding $\vec{TU}$

$$\vec{TU} = \vec{PU} - \vec{PT}$$

$$\vec{TU} = \left(\frac{u}{3}\right) - \left(\frac{v}{2} + \frac{u}{2}\right)$$

$$\vec{TU} = -\frac{\mathbf{v}}{\mathbf{2}} - \frac{\mathbf{u}}{\mathbf{6}}$$

2.8.10 Translation Vector

Activity 2.8.10 ***Work in groups***
What you require: Graph paper

(a) Draw the x axis and y axis on the graph paper.

(b) Plot the triangle with vertices $A(-3, 1)$, $B(-1, 1)$, and $C(-2, 3)$.

(c) Translate each point by moving 2 units to the right parallel to the x axis and 3 units up in the y axis. Label the new points as A', B', and C'.

(d) Draw the new triangle $A'B'C'$ on the graph.

(e) Use dotted lines to connect each original point to its corresponding translated point (A to A', B to B', C to C'), add arrows to indicate the direction.

(f) Observe and describe any similarities between triangle ABC and triangle $A'B'C'$.

(g) Analyze the distance each point moved.

(h) Discuss and share your findings with your classmates in the class.

Key Takeaway

A square **ABCD** undergoes a translation when each of its vertices (**A**,**B**,**C** and **D**) is moved the same distance and in the same direction. A translation vector, denoted by **T**, describes this movement.

Using **T** to represent a translation, the notation **T**(**P**) indicates the application of the translation **T** on **P**. In Figure 2.8.52, shows **A′B′C′D′** is the image of **ABCD** under a translation.

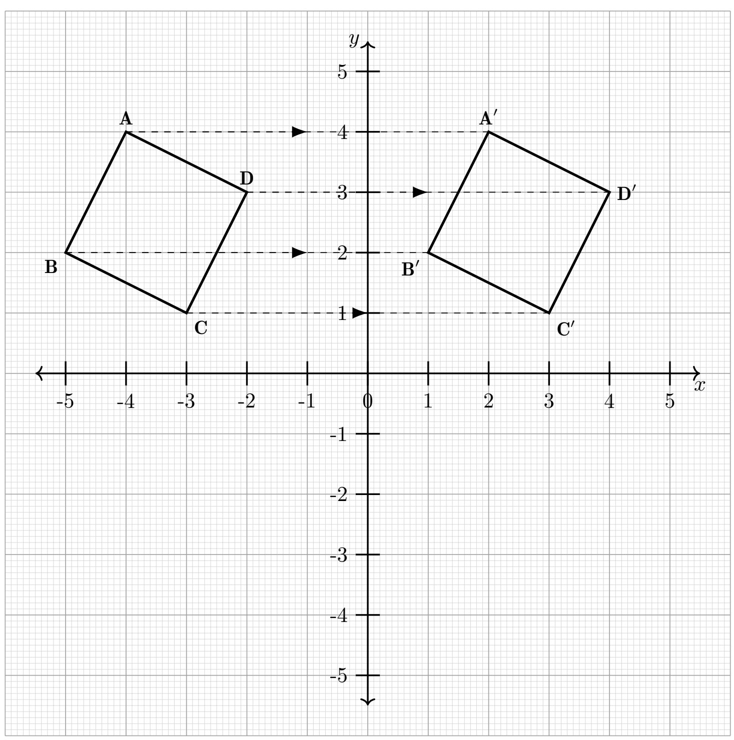


Figure 2.8.52

Example 2.8.53 Triangle **ABC** has vertices $\mathbf{A}(1,3)$, $\mathbf{B}(3,0)$ and $\mathbf{C}(4,4)$.The triangle undergoes a translation **T** defined by the vector $\begin{pmatrix}4\\3\end{pmatrix}$.

(a) Determine the coordinates of the translated vertices $\mathbf{A}'$, $\mathbf{B}'$, and $\mathbf{C}'$ '.

(b) Draw the triangle **ABC** and its image under **T**.

Solution. To find the coordinates of the translated vertices, we apply the translation **T** to each original vertex.

$$\mathbf{OA}' = \mathbf{OA} + \mathbf{T}$$
$$\mathbf{OB}' = \mathbf{OB} + \mathbf{T}$$
$$\mathbf{OC}' = \mathbf{OC} + \mathbf{T}$$

$$\mathbf{OA}' = \begin{pmatrix}1\\3\end{pmatrix} + \begin{pmatrix}4\\3\end{pmatrix} = \begin{pmatrix}1+4\\3+3\end{pmatrix} = \begin{pmatrix}5\\6\end{pmatrix}$$
$$\mathbf{OB}' = \begin{pmatrix}3\\0\end{pmatrix} + \begin{pmatrix}4\\3\end{pmatrix} = \begin{pmatrix}3+4\\0+3\end{pmatrix} = \begin{pmatrix}7\\3\end{pmatrix}$$
$$\mathbf{OC}' = \begin{pmatrix}4\\4\end{pmatrix} + \begin{pmatrix}4\\3\end{pmatrix} = \begin{pmatrix}4+4\\4+3\end{pmatrix} = \begin{pmatrix}8\\7\end{pmatrix}$$

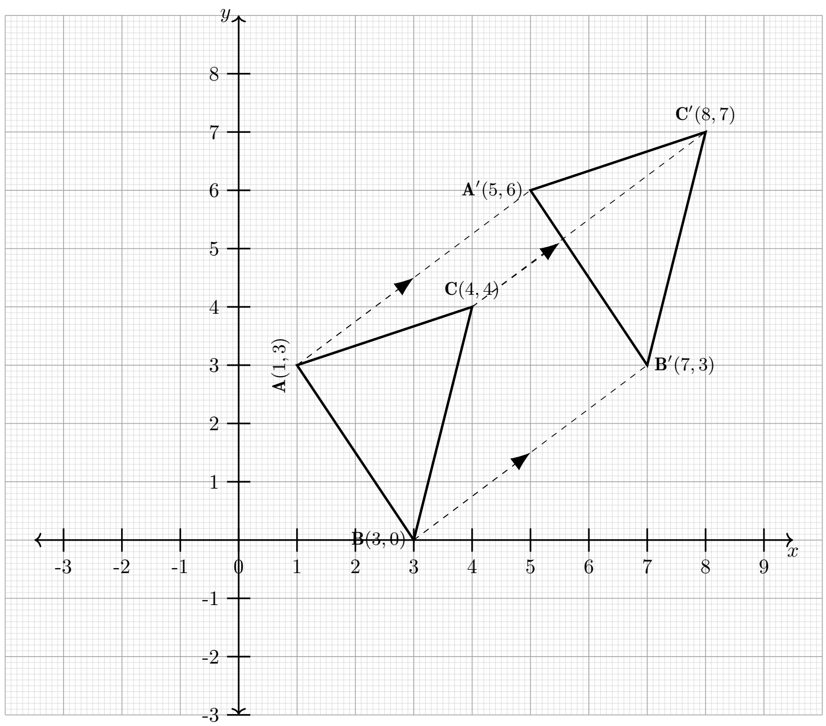


Figure 2.8.54

□

Exercises

1. Draw triangle **XYZ** with vertices $\mathbf{X(1,4)}$, $\mathbf{Y(6,2)}$, and $\mathbf{Z(5,3)}$. On the same axes, plot $\mathbf{X'Y'Z'}$, the image of triangle **XYZ** under a translation given by $\binom{4}{9}$.

2. The following points have been translated using the given vectors. Determine their original positions:

(a) $(4,-1)$; $\begin{pmatrix}2\\3\end{pmatrix}$ (c) $(-3,8)$; $\begin{pmatrix}-2\\7\end{pmatrix}$ (e) $(-12,5)$; $\begin{pmatrix}3\\-10\end{pmatrix}$

(b) $(0,-3)$; $\begin{pmatrix}1\\3\end{pmatrix}$ (d) $(11,-5)$; $\begin{pmatrix}5\\-1\end{pmatrix}$ (f) $(2,-7)$; $\begin{pmatrix}-10\\15\end{pmatrix}$

3. A point $\mathbf{P(5,-3)}$ is mapped to a new position after a translation. If the new coordinates are $\mathbf{(9,1)}$, determine the translation vector used.

4. A point $\mathbf{M(1,-4)}$ undergoes a translation by $\binom{3}{5}$. Determine the coordinates of $\mathbf{M'}$, the transformed point. If $\mathbf{M'}$ is then translated by $\binom{-4}{2}$, find the final position $\mathbf{M''}$. What is the single translation vector that maps $\mathbf{M'}$ to $\mathbf{M''}$ directly?

5. Translate each of the following points using the given vector:

(a) $(10,1)$; $\begin{pmatrix}-11\\2\end{pmatrix}$ (e) $(1,10)$; $\begin{pmatrix}7\\2\end{pmatrix}$

(b) $(-2,-5)$; $\begin{pmatrix}6\\14\end{pmatrix}$ (f) $(4,-9)$; $\begin{pmatrix}5\\-8\end{pmatrix}$

(c) $(3,-15)$; $\begin{pmatrix}-16\\11\end{pmatrix}$ (g) $(-2,13)$; $\begin{pmatrix}-3\\1\end{pmatrix}$

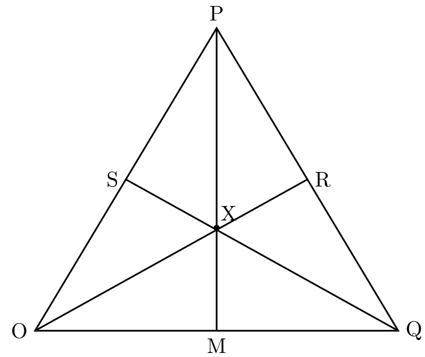
(d) $(-11,4)$; $\begin{pmatrix}-15\\10\end{pmatrix}$ (h) $(-6,5)$; $\begin{pmatrix}-1\\2\end{pmatrix}$

Further Exercises

1. Given that $a = (-3,2)$, $b = (6,-4)$ and $c = (5,-15)$and that $q = 2a + \frac{1}{2}b + \frac{1}{5}c$. Express q as a column vector and hence calculate its magnitude $|q|$ correct to two decimal places

2. If P, Q and R are the points $(2,-4)$, $(4,0)$ and $(1,6)$ respectively, use the vector method to find the coordinates of point S given that $PQRS$ is a parallelogram.

3. The figure below shows a triangle of vectors in which $OS : SP = 1 : 3$, $PR : RQ = 2 : 1$ and X is the midpoint of OR.



Given that $OP = p$ and $OQ = q$, express the following vectors in terms of p and q

(i) PQ (ii) QR (iii) OR (iv) QT

4. Given that $3p - 2q = \binom{10}{5}$ and $p + 2q = \binom{-7}{15}$ find;

(a) p and q (b) $|2p + 3q|$

5. In triangle OAB, M and N are points on OA and OB respectively, such that $OM : OA = 2 : 5$ and $ON : OB = 2 : 3$. AN and BM intersect at T. Given that $OA = a$ and $OB = b$.Express in terms of a and b:

(a). AB (b) BM (C) AN

6. Find the scalars p and q
$p\binom{8}{6} + q\binom{-6}{4} = \binom{10}{16}$

7. A point M divides PQ in the ratio 3 : 5. Given that P is $(-6, 8)$ and Q is $(4, -2)$. Find the coordinates of M

2.9 Linear Motion

Linear motion refers to the movement of an object in a straight line.

It is described using key concepts such as distance, displacement, speed, velocity and acceleration.

Linear motion is present in many real-world scenarios:

i. Vehicles - Cars accelerating, stopping, and moving at constant speed.

ii. Sports - Running, cycling, and swimming events involve linear motion.

iii. Free Fall - Objects falling under gravity move in linear motion.

2.9.1 Speed

Activity 2.9.1 ***Work in groups***

Materials: Stop watch, Measuring tape and ball/round object

i). Roll a ball or any round object on the table

ii). Measure the rolling distance using the tape measure

iii). Record the time when rolling

iv). Find the speed of the rolling ball

Activity 2.9.2 ***Work in groups***

A family drives 120 km at 60 km/h, stops for a 15-minute break, and then drives another 80 km at 40 km/h. What is the average speed for the entire trip?

1. Calculate the time for each driving segment
2. Convert the break time to hours
3. Calculate the total distance
4. Calculate the total time
5. Calculate the average speed

Key Takeaway 2.9.1 **Speed** is the rate of change of distance with time. SI unit: km/h .

Speed normally varies over time, so the average speed is often used

$$\text{Average speed} = \frac{\text{Total distance covered}}{\text{Total time taken}}$$

Example 2.9.2 A car travels 100 km in 2 hours. Find its speed.

Solution. Speed = $\frac{\text{distance}}{\text{time}}$

Speed = $\frac{100}{2}$ = 50 km/h □

Example 2.9.3 A motorist covers a distance of 360 km in 3 hours travelling from busia to bahangongo. Calculate the average speed.

Solution. Speed = $\frac{\text{distance}}{\text{time}}$

$$\text{Average speed} = \frac{360 \text{ km}}{3 \text{ h}}$$

$$120 \text{ km/h}$$

□

Example 2.9.4 Alex is cycling from home to the market. He rides the first 8 km at a speed of 16 km/h. After reaching a park, Alex takes a 20-minute break to play football with his friends. Then, he cycles the remaining 6 km to the market at a speed of 12 km/h. What was Alex's average speed for the entire journey?

Solution.

1. Calculating the time for each cycling part.

   - ***part 1***
     Distance: $8km$
     Speed: $16km/h$

     $$\text{Time } = \frac{\text{Distance}}{\text{Speed}} = \frac{8 \text{ km}}{16 \text{ km/h}} = 0.5 \text{ hours}$$

   - ***part 2***
     Distance: $6km$

Speed: $12km/h$

$$\text{Time} = \frac{\text{Distance}}{\text{Speed}} = \frac{6\text{ km}}{12\text{ km/h}} = 0.5\text{ hours}$$

2. Convert the break time to hours
   $\frac{20\text{ minutes}}{60\text{ minutes}} = \frac{1}{3}$ hours
3. Calculating the total distance gives
   $$\text{Total distance} = 8\text{ km} + 6\text{ km} = 14\text{ km}$$
4. Calculating the total time gives
   Total time =
   $$\frac{1}{2}\text{ hours} + \frac{1}{3}\text{ hours} + \frac{1}{2}\text{ hours} = 1\frac{1}{3}\text{ hours}$$
5. Now Calculating the average speed gives
   Average speed $= \frac{\text{Total distance}}{\text{Total time}}$
   Average speed $= \frac{14\text{ km}}{1.33\text{ hours}} = 10.53$ km/h

Alex's average speed for the entire journey was approximately 10.53 km/h.

☐

Checkpoint 2.9.5 A cyclist travels at 25 km/h for 2 hours, then at 45 km/h for 9 hours.

Calculate the average speed for the entire journey.

Average speed = ______ km/h

Answer. $\frac{455}{11}$

Solution. Worked Solution ***Step 1: Calculate distance for each segment*** First segment: Distance = Speed × Time = 25 × 2 = 50 km Second segment: Distance = Speed × Time = 45 × 9 = 405 km

Step 2: Calculate total distance and total time Total distance = 50 + 405 = 455 km Total time = 2 + 9 = 11 hours

Step 3: Calculate average speed Therefore, average speed is given by:

$$\begin{aligned} &= \frac{\text{Total distance}}{\text{Total time}} \\ &= \frac{455}{11} \\ &= \frac{455}{11}\text{ km/h} \end{aligned}$$

Checkpoint 2.9.6 A car travels 300 km at 70 km/h, then 300 km at 40 km/h.

Calculate the average speed for the entire 600 km journey.

Average speed = ______ km/h

Answer. $\frac{560}{11}$

Solution. Worked Solution ***Step 1: Calculate time for each segment*** First segment: Time = $\frac{300}{70} = \frac{30}{7}$ hours Second segment: Time = $\frac{300}{40} = \frac{15}{2}$ hours

Step 2: Calculate total time and total distance Total distance = 300

+ 300 = 600 km Total time $= \frac{30}{7} + \frac{15}{2} = \frac{165}{14}$ hours

Step 3: Calculate average speed Average speed $= \frac{\text{Total distance}}{\text{Total time}} = \frac{600}{\frac{165}{14}} = \frac{560}{11}$ km/h

Checkpoint 2.9.7 A bus travels 480 km in 2 hours. Calculate its speed.

Speed = ______ km/h

Answer. 240

Solution. Worked Solution By definition speed = Distance ÷ Time

Speed $= \frac{480 \text{ km}}{2 \text{ hours}} = 240$ km/h

Therefore, the bus travels at a speed of 240 km/h

Exercises

1. Juma cycled for 3 hours to a trading centre 60 km away, then drove 250 km in a van at 50 km/h and finally cycled home at 20 km/h. Find the average speed for the whole journey.

Answer. Average speed $= \frac{60+250+60}{3+\frac{250}{50}+\frac{60}{20}}$

$= \frac{370}{11}$ km/h

2. Sarah is rushing to school. She walks the first 500 meters at 5 km/h, realizing she's late, she then sprints the next 300 meters at 10 km/h What was her average speed for the entire trip to school?

Answer. Average speed $= \frac{0.5+0.3}{\frac{0.5}{5}+\frac{0.3}{10}}$

$= \frac{0.8}{0.1+0.03} = \frac{0.8}{0.13} = \frac{80}{13}$ km/h

3. John walks to the store, a distance of 1 km, at a speed of 4 km/hr. He spends 15 minutes at the store, and then walks back at 5 km/hr. What was John's average speed for the entire trip, including the time spent at the store

Answer. Average speed $= \frac{1+1}{\frac{1}{4}+\frac{1}{5}+\frac{15}{60}}$

$= \frac{2}{0.25+0.2+0.25} = \frac{2}{0.7} = \frac{20}{7}$ km/h

4. A hiker covered 12 km in 3 hours. After taking a 15 minute break, what speed must the hiker maintain to reach their destination within a total travel time of 4 hours?

Answer. Required speed $= \frac{12 \text{ km}}{4 \text{ hours} - 3 \text{ hours} - \frac{15}{60} \text{hours}}$

$= \frac{12 \text{ km}}{0.75 \text{ hours}} = 16$ km/h

5. A runner completed 10 km in1.5 hours. After a 5 minute rest, what pace does the runner need to maintain to finish a total distance within 2 hours?

Answer. Required pace $= \frac{10 \text{ km}}{2 \text{ hours} - 1.5 \text{ hours} - \frac{5}{60} \text{hours}}$

$= \frac{10 \text{ km}}{0.5 - \frac{5}{60}} = \frac{10}{\frac{5}{60}} = 24$ km/h

6. A train journey consisted of three segments. The train traveled for 3 hours at 90 km/h, then paused for 0.75 hours at a station, and finally continued for 2 hours at 70 km/h. What was the average speed of the train for the entire journey?

Answer. Average speed $= \frac{(3\times 90)+(2\times 70)}{3+0.75+2}$

$= \frac{270+140}{5.75} = \frac{410}{5.75} = \frac{1640}{23}$ km/h

2.9.2 Velocity and Acceleration in Different Situations

Activity 2.9.3 Work in groups

1. Define
   a). Velocity
   b). Velocity
2. A car travels 200 meters in 10 seconds. What is its average velocity?
3. A car accelerates from 75 km/h to 90 km/h in 10 seconds.
   Find it's acceleration
4. A vehicle moving at 25 m/s applies brakes and comes to a stop in 5 seconds.
   a). What is the acceleration of the vehicle?
   b). Is the acceleration positive or negative? Why?

Key Takeaway 2.9.8 Velocity is Speed in a specified direction or the rate of change of displacement with time.

Symbol of velocity is given as **v** while speed is given as **s**.
$v = \frac{d}{t}$
where d represents the distance and t represents time
For motion with **constant velocity**, the equation is

$$s = vt$$

where v is velocity , t is time and s is displacement
Acceleration - The rate of change of velocity with time. SI unit is m/s^2
Acceleration is given by

$$\text{Acceleration } = \frac{\text{Change in velocity}}{\text{Time Taken}}$$

$$a = \frac{\Delta v}{\Delta t}$$

Negative acceleration is called **deceleration or retardation**

For motion with **constant acceleration**, the three equations of motion are:

i). final velocity = Initial velocity + (acceleration × time)
$v = u + at$
where;
- v is Final velocity
- u is initial velocity
- a is acceleration
- t is time

ii). Displacement = (initial velocity × time) + ($\frac{1}{2}$ × acceleration × time^2)
$s = ut + \frac{1}{2}at^2$

iii). Final velocity^2 = Initial velocity^2 + (2 × acceleration × displacement)
$v^2 = u^2 + (2 \times \textbf{acceleration}) \times s^2$

Example 2.9.9 A car starts from rest and accelerates at 2 m/s^2 for 5 seconds. Find its final velocity.

Solution. $v = u + at$
$= 0 + (2 \times 5) = 10$ m/s
The final velocity is 10 m/s □

Example 2.9.10 A rocket accelerates from rest to 250 m/s in 10 seconds. Calculate the rocket's acceleration.

Solution. Initial velocity $u = 0$m/s
Final velocity $v = 250$m/s
Time (t) = 10seconds
Acceleration (a) = (v - u) / t

$$\frac{(v-u)}{t}$$

$$a = \frac{(250-0)\,\mathrm{m/s}}{10s}$$

$$a = \frac{(250\,\mathrm{m/s})}{10\,\mathrm{s}}$$

$$a = 25\,\mathrm{m/s}^2$$

The rocket's acceleration is $25\,\mathrm{m/s}^2$ □

Example 2.9.11 An object decelerates from 20 m/s to 5 m/s in 3 seconds. What is its acceleration?

Solution. Initial velocity $u = 20$m/s
Final velocity $v = 5$m/s
Time (t) = 3seconds
Acceleration (a) = (v - u) / t

$$\frac{(v-u)}{t}$$

$$a = \frac{(5-20)\,\mathrm{m/s}}{3\,\mathrm{seconds}}$$

$$a = \frac{(-15\,\mathrm{m/s})}{3\,\mathrm{seconds}}$$

$$a = -5\,\mathrm{m/s}^2$$

The object's acceleration is $-5\,\mathrm{m/s}^2$. The negative sign indicates deceleration (slowing down). □

Exercises

1. A car's velocity changes from $10\,\mathrm{m/s}$ to $30\,\mathrm{m/s}$ in 4 seconds. Find its acceleration.

Answer. $a = \frac{(30\,\mathrm{m/s} - 10\,\mathrm{m/s})}{4\,\mathrm{s}} = \frac{20\,\mathrm{m/s}}{4\,\mathrm{s}} = 5\,\mathrm{m/s}^2$

2. A car moves with $8\,\mathrm{m/s}^2$ acceleration for 5 seconds, reaching $40\,\mathrm{m/s}$. Find its initial velocity.

Answer. $u = v - at = 40\,\mathrm{m/s} - (8\,\mathrm{m/s}^2 \times 5\,\mathrm{s}) = 40\,\mathrm{m/s} - 40\,\mathrm{m/s} = 0\,\mathrm{m/s}$

3. A train moving at $40\,\mathrm{km/h}$ decelerates at $0.5\,\mathrm{m/s}^2$. Find the time taken to stop

Answer. $t = \frac{(0\,\mathrm{m/s} - 11.11\,\mathrm{m/s})}{-0.5\,\mathrm{m/s}^2} = \frac{-11.11\,\mathrm{m/s}}{-0.5\,\mathrm{m/s}^2} = 22.22\,s$

4. A cyclist's speed increases from 5 m/s to 17 m/s over a period of 6 seconds. What is the cyclist's average acceleration?

Answer. $a = \frac{(17\,\mathrm{m/s} - 5\,\mathrm{m/s})}{6\,\mathrm{s}} = \frac{12\,\mathrm{m/s}}{6\,\mathrm{s}} = 2\,\mathrm{m/s}^2$

5. A runner's velocity changes from 3 m/s to 7 m/s. What is the runner's average velocity?

Answer. Average velocity $= \frac{(3\,\mathrm{m/s} + 7\,\mathrm{m/s})}{2} = \frac{10\,\mathrm{m/s}}{2} = 5\,\mathrm{m/s}$

Checkpoint 2.9.12 A car starts from rest and accelerates uniformly at 7 m/s² for 4 seconds.

a) What is its final velocity? Final velocity = ______ m/s

b) How far does it travel during this time? Distance = ______ m

Answer 1. 28

Answer 2. 56

Solution. Worked Solution ***Part (a): Final velocity*** Using $v = u + at$ Initial velocity $u = 0$ m/s (starts from rest) Acceleration $a = 7$ m/s² Time $t = 4$ s $v = 0 + (7) \times (4) = 28$ m/s

Part (b): Distance traveled Using $s = ut + \frac{1}{2}at^2$ $s = 0 \times 4 + \frac{1}{2} \times 7 \times (4)^2$ $s = \frac{1}{2} \times 7 \times 16 = 56$ m

Checkpoint 2.9.13 A car traveling at 120 m/s brakes uniformly and comes to rest in 10 seconds.

a) What is its deceleration? Deceleration = ______ m/s²

b) How far does it travel before stopping? Stopping distance = ______ m

Answer 1. -12

Answer 2. 600

Solution. Worked Solution Given: $u = 120$ m/s, $v = 0$ m/s, $t = 10$ s

Part (a): Deceleration

$$\begin{aligned} a =& \frac{v-u}{t} \\ =& \frac{0-120}{10} \\ =& -12 \text{ m/s}^2 \end{aligned}$$

(negative indicates deceleration) ***Part (b): Distance travelled before stopping***

$$s = \frac{u+v}{2} \times t$$

$$= \frac{120 + 0}{2} \times 10$$
$$= 600 \text{ m}$$

2.9.3 Displacement Time Graph of Different Situations

Activity 2.9.4 Work in groups

A motorist travels from Limuru to Kisumu. The table below shows the distances covered at different times:

Time	Distance (km)
9:00 AM	0
10:00 AM	80
11:00 AM	160
11:30 AM	160
12:00 PM	210

Plot the graph using the data given in the table and use it to answer the questions below

a). How far was the motorist from Limuru at 10:30 AM?

b). What was the average speed during the first part of the journey?

c). What was the overall average speed?

Key Takeaway 2.9.14 Distance is the total length of the path traveled by an object.

Displacement is the shortest distance from the initial to the final position of an object, represented as a vector.

When distance is plotted against time, a distance-time graph is obtained.

Example 2.9.15 A car moves with a constant velocity of 5 m/s for 8 seconds.

Draw the displacement-time graph and determine the displacement at t = 6s.

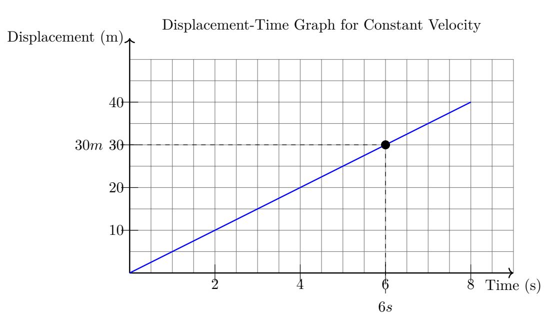
Solution. Since the velocity is constant, the displacement increases linearly with time

Hence ,

$$s = vt$$

at t = 6s

$$S = 5 \times 6 = 30$$



□

Example 2.9.16 A car starts from rest and accelerates uniformly at $2\,\mathrm{m/s^2}$ for 5 seconds.

Draw the displacement-time graph.

Solution. Since acceleration is constant, the displacement follows the equation

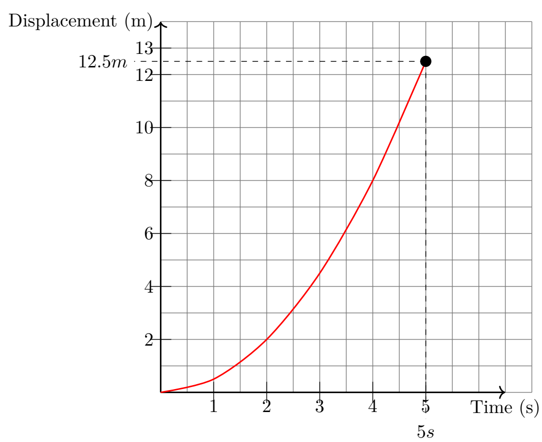
$$s = \frac{1}{2}at^2$$

for different time values:

Time (s)	Displacement (m)
1	0.5
2	2
3	4.5
4	8
5	12.5

Here is the graph

Displacement-Time Graph for Uniform Acceleration



□

Example 2.9.17 A car moves in three different phases as shown below;

The car starts from rest and accelerates uniformly. The car moves at a constant velocity. The car comes to a stop and remains at a fixed position.

a). Sketch a displacement-time graph for the motion.

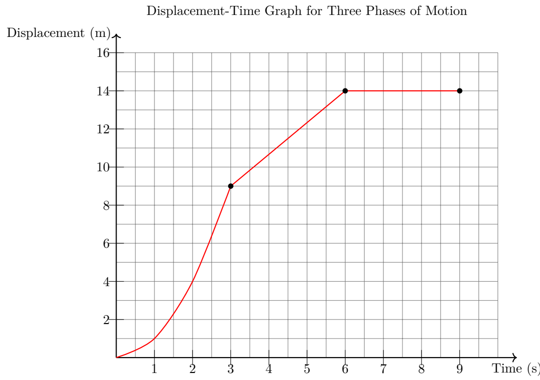
b). Identify the type of motion in each phase.

c). Determine the displacement at t = 3s, t = 6s, and t = 9s .

Solution. 0s to 3s - The displacement follows a curved path because the car is accelerating.

3s to 6s - The displacement increases linearly since the velocity is constant.

6s to 9s - The displacement remains constant because the car has stopped.



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Exercises

1. A car moves along a straight road, and its displacement from the starting point is recorded at different times.
The data is shown below;

Time (s)	Displacement (m)
0	0
2	4
4	10
6	16
8	20
10	20
12	15
14	8
16	0

(a) Plot a displacement-time graph using the data above

(b) Describe the motion of the car based on the graph.

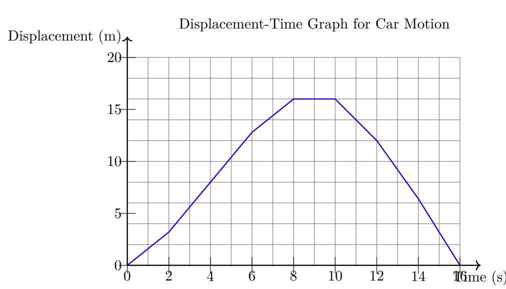
(c) Identify the time intervals when the car is at rest

(d) Find the velocity of the car at the following intervals

- 0 to 6 seconds
- 6 to 10 seconds
- 10 to 16 seconds

(e) Determine the total distance traveled by the car.

Answer.



(a) The car accelerates from rest to a displacement of 20 m in the first 8 seconds.

The car remains at rest from 8 to 10 seconds.

The car then decelerates back to the starting point over the next 6 seconds.

(b) The car is at rest between 8 to 10 seconds.

(c)
- From 0 to 6 seconds: Velocity $= \frac{16\,\text{m}}{6\,\text{s}} = 2.67\,\text{m/s}$
- From 6 to 10 seconds: Velocity $= 0\,\text{m/s}$ (car is at rest)
- From 10 to 16 seconds: Velocity $= \frac{-15\,\text{m}}{6\,\text{s}} = -2.5\,\text{m/s}$

(d) The total distance traveled by the car is 40 m (20 m forward and 20 m backward).

2. Study the following description of a runner's motion and sketch the corresponding displacement-time graph

- The runner starts from rest and accelerates uniformly for 5 seconds, covering a displacement of 25 meters.
- The runner maintains a constant speed for the next 10 seconds, covering an additional 50 metres.
- The runner then decelerates uniformly for 5 seconds until stopping at 100 meteres.

(a) Sketch the displacement-time graph based on this motion.

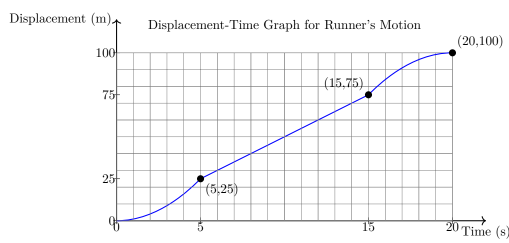
(b) Determine the velocity during the constant speed phase.

(c) Calculate the acceleration during the first 5 seconds.

(d) Find the total time taken to complete the journey.

(e) What is the average velocity for the entire motion?

Answer.



(a) The velocity during the constant speed phase is $\frac{50\,\text{m}}{10\,\text{s}} = 5\,\text{m/s}$.

(b) The acceleration during the first 5 seconds is $\frac{25\,\text{m}}{(5\,\text{s})^2} = 2\,\text{m/s}^2$.

(c) The total time taken to complete the journey is $5 + 10 + 5 = 20\,\text{seconds}$.

(d) The average velocity for the entire motion is $\frac{100\,\text{m}}{20\,\text{s}} = 5\,\text{m/s}$.

3. The displacement-time graph represents the motion of a cyclist

- From 0 to 4 seconds, the cyclist moves forward at a uniform velocity.

- From 4 to 8 seconds, the cyclist is stationary.
- From 8 to 12 seconds, the cyclist moves back towards the starting point at a uniform velocity.

(a) Sketch the graph for this motion.

(b) What is the velocity during the first 4 seconds?

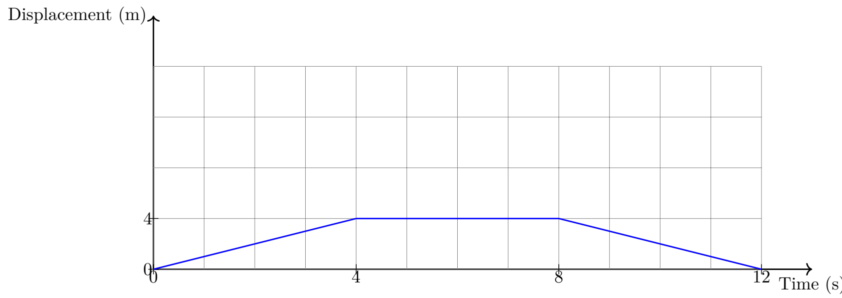
(c) What does the flat section of the graph indicate?

(d) Find the velocity during the last 4 seconds.

(e) Calculate the total displacement at the end of 12 seconds.

Answer.

Displacement-Time Graph for Cyclist's Motion



(a) The velocity during the first 4 seconds is $\frac{4\,\mathrm{m}}{4\,\mathrm{s}} = 1\,\mathrm{m/s}$.

(b) The flat section of the graph indicates that the cyclist is stationary during that time interval.

(c) The velocity during the last 4 seconds is $\frac{-4\,\mathrm{m}}{4\,\mathrm{s}} = -1\,\mathrm{m/s}$.

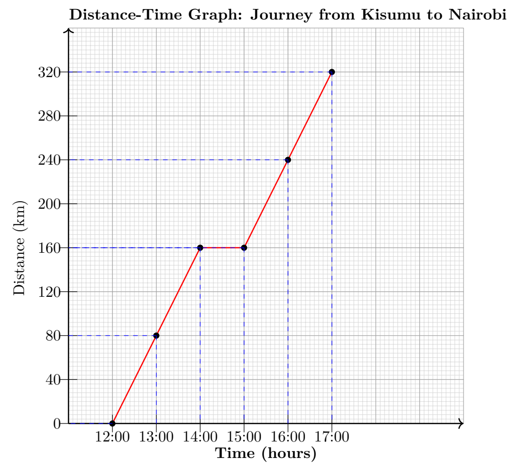
(d) The total displacement at the end of 12 seconds is 0 meters.

Checkpoint 2.9.18 This question contains interactive elements.

2.9.4 Interpretation of Displacement Time Graph

Activity 2.9.5 ***Work in groups***

1. Consider the displacement time graph representing distance covered by a motorist traveling from Turkana to Nairobi.



2. How far was the motorist from the starting point at $2:30$ PM?

3. What was the total distance covered by the motorist?

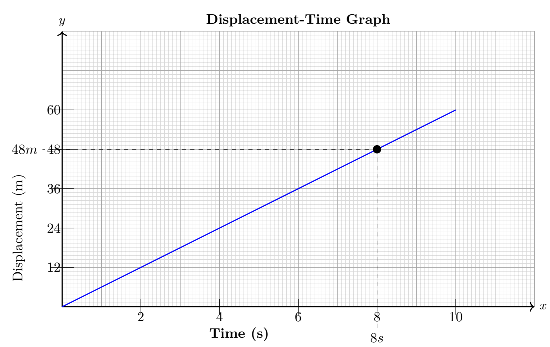
4. During which periods was the motorist stationary?

5. Calculate the average speed of the motorist between 12 noon and 2 PM.

6. What was the overall average speed for the entire journey?

Key Takeaway 2.9.19 A displacement-time graph shows how the displacement of an object changes over time. The slope of the graph at any point indicates the velocity of the object at that moment. A straight line with a constant slope represents uniform motion, while a curved line indicates acceleration or deceleration.

Example 2.9.20 Use the displacement-time graph for constant velocity to answer the following questions.



a) What type of motion does the graph represent? Explain your answer.

b) What is the displacement of the car at $t = 8$ seconds?

c) What is the total displacement at $t = 10$ seconds

d) Determine the velocity of the car from the graph.

Solution.

a) The graph shows a straight line with a constant slope, indicating uniform motion. This means the car is moving at a constant velocity with no acceleration.

b) From the graph, the displacement at $t = 8$ seconds is 48 meters.

c) Using the equation of motion:

$$
\begin{aligned}
s =& t \times v \\
s =& 6 \times 10 \\
=& 60 \, \text{meters}
\end{aligned}
$$

Therefore, $t = 10$ seconds, the car's displacement is 60 meters.

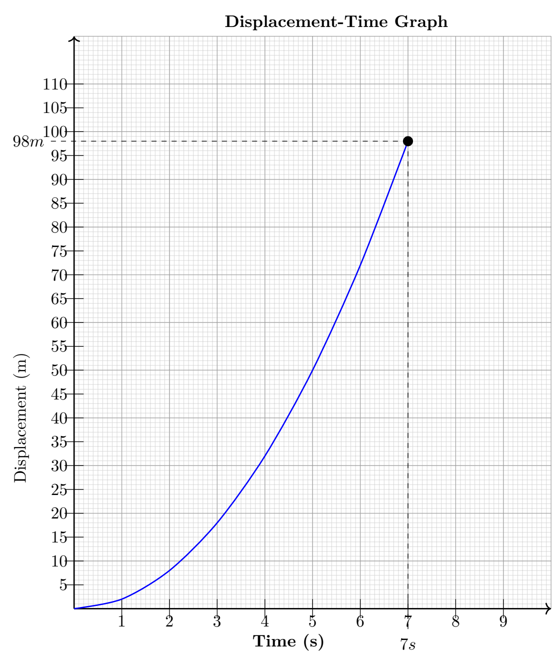
d) Velocity is given by the slope of the displacement-time graph:

$$
\begin{aligned}
\text{Velocity} =& \frac{\text{Change in displacement}}{\text{Change in time}} \\
\text{Velocity} =& \frac{48 - 0}{8 - 0} \\
=& \frac{48}{8} = 6
\end{aligned}
$$

This confirms the car's velocity is 6 m/s.

□

Example 2.9.21 Use the graph below to answer the questions.



a) What type of motion is represented by the graph? Explain your reasoning.

b) If the train continued to accelerate at 4 m/s², what would be its approximate displacement at $t = 8$ seconds?

c) What is the displacement of the body at $t = 7$ seconds?

Solution.

a) The graph represents uniformly accelerated motion.

The line on the graph is curved, not straight. This means the object is not moving the same distance each second. Its speed is changing.

b) Using the equation:

$$S = \frac{1}{2}at^2 + ut$$

$$S = \frac{1}{2}4(8)^2 + (0)(8)$$

$$S = 0 + 2(64)$$

$$S = 128 \text{ m}$$

The approximate displacement at $t = 8$ seconds would be 128 meters.

c) 98 m.

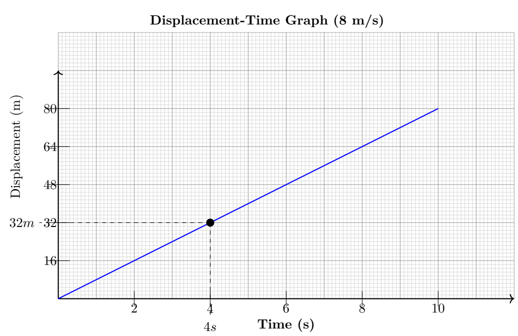
□

Checkpoint 2.9.22 This question contains interactive elements.

Checkpoint 2.9.23 This question contains interactive elements.

Exercises

1. The displacement-time graph below shows a train moving at a constant velocity.



Use the graph to answer the following questions.

a) What is the displacement at $t = 4\,\text{s}$?

b) If the object continued moving for 15 seconds, what would be its total displacement?

c) What would the graph look like if the object stopped moving after 6 seconds?

Answer.

a) From the graph, at $t = 4\,\text{s}$, the displacement is $32\,\text{m}$.

b) The object moves at a constant velocity of $8\,\text{m/s}$. Therefore, in $15\,\text{s}$, the total displacement would be:

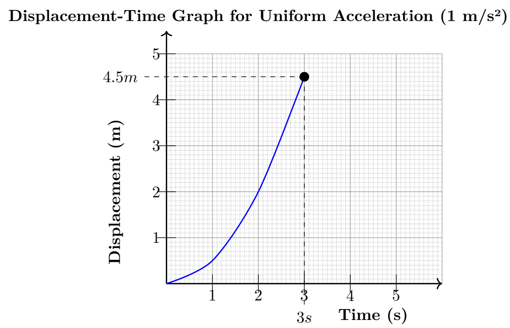
$$s = v \times t$$

$$s = 8\,\text{m/s} \times 15\,\text{s}$$
$$s = 120\,\text{m}$$

Thus, the total displacement after 15 s would be 120 m.

c) If the object stopped moving after 6 s, the graph would show a horizontal line starting from the point corresponding to $t = 6\,\text{s}$ and the displacement at that time. This indicates that the displacement remains constant after that point.

2. The displacement-time graph below shows a car accelerating smoothly from rest.



Use the graph to answer the following questions:

a) Describe the motion of the car as shown in the graph. Is the velocity constant, increasing, or decreasing? Justify your answer.

b) What does the y-intercept of the graph represent?

c) Calculate the average velocity of the object between **t** = 0 **s** and **t** = 3 **s**

Answer.

a) The graph shows a curved line that gets steeper over time, indicating that the car's velocity is increasing. This means the car is accelerating.

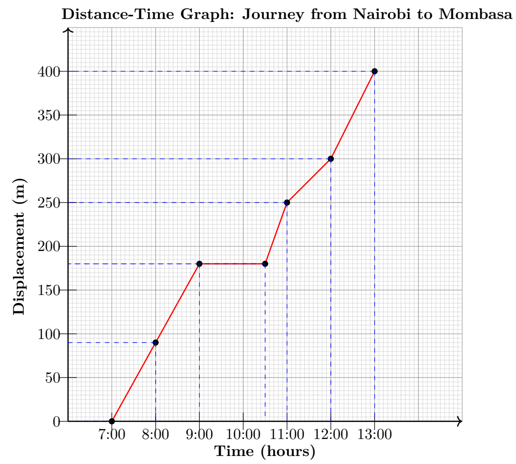
b) The y-intercept of the graph represents the initial displacement of the car at time $t = 0\,\text{s}$. In this case, it is 0 m, indicating that the car started from rest at the origin.

c) The average velocity can be calculated using the formula:

$$\begin{aligned}\text{Average Velocity} &= \frac{\text{Total Displacement}}{\text{Total Time}}\\ &= \frac{4.5\,\text{m} - 0\,\text{m}}{3\,\text{s} - 0\,\text{s}}\\ &= \frac{4.5\,\text{m}}{3\,\text{s}} = 1.5\,\text{m/s}\end{aligned}$$

Therefore, the average velocity of the car between $t = 0\,\text{s}$ and $t = 3\,\text{s}$ is 1.5 m/s.

3. The distance-time graph below shows a motorist traveling from Nairobi to Mombasa with varying speeds and periods of rest.



Use the graph to answer the following questions:

a) What was the total distance traveled by the motorist?

b) At what time did the motorist stop for a break?

c) Calculate the speed of the motorist between 7 : 00 **AM** and 8 : 00 **AM**.

d) What was the speed of the motorist from 9 : 00 **AM** and 10 : 30 **AM**?

e) Find the overall average speed of the entire journey.

f) Identify a section of the graph where the motorist was stationary.

g) Describe what happens when the graph has a steeper slope.

Answer.

a) The total distance traveled by the motorist is 400 km.

b) The motorist stopped for a break between 9 : 00 AM and 10 : 30 AM.

c) The speed of the motorist between 7 : 00 AM and 8 : 00 AM is:

$$\text{Speed} = \frac{\text{Distance}}{\text{Time}} = \frac{90 \text{ km}}{1 \text{ hour}} = 90 \text{ km/h}$$

d) The speed of the motorist from 9 : 00 AM to 10 : 30 AM is:

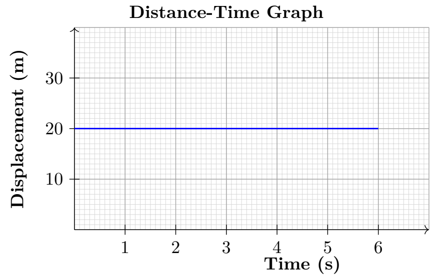
$$\text{Speed} = \frac{\text{Distance}}{\text{Time}} = \frac{0 \text{ km}}{1.5 \text{ hours}} = 0 \text{ km/h}$$

e) The overall average speed of the entire journey is:
Average Speed $= \frac{400 \text{ km}}{6 \text{ hours}} = 66.67$ km/h

f) The motorist was stationary between 9 : 00 AM and 10 : 30 AM.

g) A steeper slope on the graph indicates a higher speed. The steeper the slope, the faster the motorist is traveling.

4. The displacement-time graph below shows a car parked on the roadside.



a) What is the displacement of the object throughout the time interval shown in the graph?

b) During which time interval(s) is the object stationary?

c) What is the total distance covered by the car in 6 seconds?

d) What is the speed of the car during the time interval shown?

Answer.

a) The displacement of the object throughout the time interval shown in the graph is 20 m.

b) The object is stationary during the entire time interval from 0 s to 6 s.

c) The total distance covered by the car in 6 s is 0 m, as it did not move.

d) The speed of the car during the time interval shown is 0 m/s, since it remained stationary.

2.9.5 Velocity Time Graph

Activity 2.9.6 ***Work in groups***

1. A train moving at 40 m/s along a North-South railway track passes through a station R at 5 : 30 PM. The train is decelerating at 4 m/s^2 northward.

   Find the velocity of the train:

   - 3seconds after 5 : 30 PM
   - 6seconds after 5 : 30 PM
   - 2seconds after 5 : 30 PM

2. Determine the average velocity of the train:

   - In the first 5 seconds after 5 : 30 PM
   - In the first 10 seconds after 5 : 30 PM

3. Draw a velocity-time graph, find the distance of the train from R at 12 seconds past 5 : 30 PM

Key Takeaway 2.9.24 A velocity-time graph is a graph that shows how an object's velocity (speed with direction) changes over time.

A velocity-time graph displays velocity on the y-axis and time on the x-axis, where the slope indicates acceleration and the area under the graph represents displacement.

A horizontal line signifies constant velocity, an upward slope indicates acceleration, a downward slope indicates deceleration, and a line at zero velocity means the object is stationary.

Acceleration is the rate of change of velocity of an object over time. It occurs when an object speeds up, slows down, or changes direction.

$$\text{Acceleration} = \frac{\text{Change in velocity}}{\text{Corresponding change in time}}$$
$$\text{a} = \frac{\Delta\text{v}}{\Delta\text{t}}$$
$$\text{a} = \frac{\text{v - u}}{\text{t}}$$

Where;

- a = Acceleration which is measured in m/s^2
- Δv = v - u = change in velocity which is measured in metres per second m/s

  v is the final velocity and u is the initial velocity.
- t = t = time taken which is measured in seconds(s)

Deceleration is negative acceleration, meaning the object is slowing down. It happens when the velocity decreases over time.

- In a velocity-time graph, deceleration appears as a downward-sloping line.

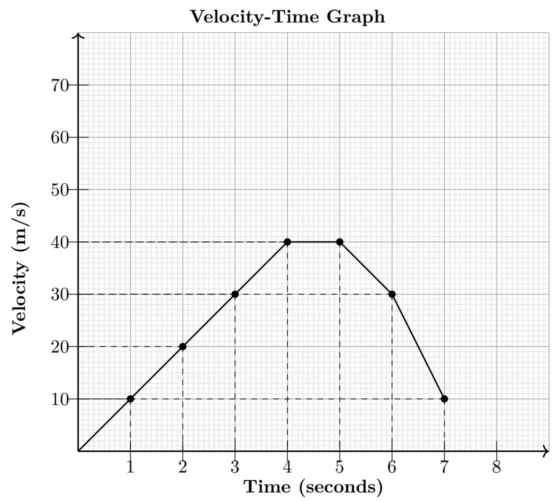
Example 2.9.25 Given the table below

Table 2.9.26

Time	0	1	2	3	4	5	6	7
Velocity (m/s)	0	10	20	30	40	40	30	10

Draw the velocity-time graph to represent the data.

Solution. This is a velocity-time graph that shows Acceleration, Constant Speed, and Deceleration.



□

Example 2.9.27 A car starts from rest and accelerates to a velocity of 40 m/s in 10 seconds . It then maintains this velocity for 15 seconds before decelerating to rest, with the total time of motion being 45 seconds

a) Draw the velocity-time graph to represent its motion.

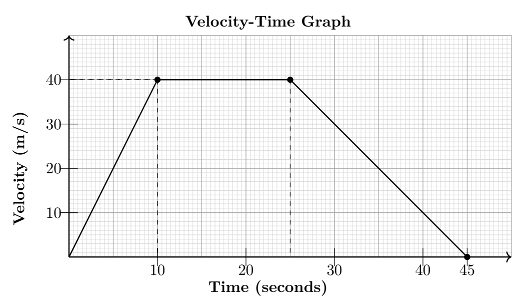
b) Find the total distance covered.

c) Determine the average velocity.

d) Calculate the acceleration and deceleration.

Solution.

a) The velocity-time graph;



b) The total distance is the area under the velocity-time graph, which consists of a trapezium.

- Distance during acceleration (triangle).

$$\begin{aligned}\text{Area } =& \frac{1}{2} \times \text{ base} \times \text{ height}\\ & \frac{1}{2} \times 10 \times 40\\ =& 200 \text{ m}\end{aligned}$$

- Distance during constant velocity (rectangle)

$$\text{Area} = \text{base} \times \ \text{height}$$
$$15 \times 40$$
$$=600 \ \text{m}$$

- Distance during deceleration (triangle)

$$\text{Area} = \frac{1}{2} \times \ \text{base} \times \ \text{height}$$
$$20 \times 40$$
$$=400 \ \text{m}$$

- Total distance:
  $200 + 600 + 400 = 1200 \ \text{m}$
  Therefore, the total distance is 1200 m

c) Average Velocity

$$\text{Average velocity} = \frac{\text{Total distance}}{\text{Total time}}$$
$$= \frac{1200}{45}$$
$$= 26.67 \ \text{m/s}$$

Therefore, the average velocity is 26.67 m/s

d) Acceleration and Deceleration
Using the equation:

$$\text{a} = \frac{\text{v - u}}{\text{t}}$$

Where;
v = is the final velocity (40 m/s, from the graph)
u = is the initial velocity (0 m/s)
t = is the time taken during acceleration (10 seconds).

$$\frac{40 - 0}{10}$$
$$=4 \ \text{m/s}^2$$

Therefore, acceleration is 4 m/s^2
Under deceleration;
v is the final velocity (0 m/s, since the car comes to rest),
u is the initial velocity (40 m/s, from the graph)
t is the time taken during deceleration (10 seconds).

$$\frac{0 - 40}{10}$$
$$= -4 \ \text{m/s}^2$$

Therefore, the object is decelerating at 4 m/s^2

□

Checkpoint 2.9.28 The velocity–time graph of a moving object shows the following motion phases:

1. Constant velocity of 40 m/s for 8 s
2. Uniform acceleration to 60 m/s over 5 s
3. Constant velocity of 60 m/s for 8 s

a) Find the acceleration during phase 2. Acceleration = ______ m/s²
b) Find the total distance travelled. Total distance = ______ m

Answer 1. 4

Answer 2. 1050

Solution. ***Worked Solution***
The motion is represented on a velocity–time graph. The total distance travelled is equal to the area under the graph.
a) Acceleration during phase 2
Acceleration is the change in velocity divided by the time taken.

$$a = \frac{\text{change in velocity}}{\text{time taken}}$$

$$a = \frac{v_2 - v_1}{t_2}$$

$$a = \frac{60 - 40}{5}$$

$$a = 4 \text{ m/s}^2$$

b) Total distance travelled
The total distance is the sum of the areas under the velocity–time graph for the three phases.

- ***Phase 1:*** Constant velocity 40 m/s for 8 s

$$\text{Distance}_1 = v_1 \times t_1 = 40 \times 8 = 320 \text{ m}$$

- ***Phase 2:*** Uniform acceleration from 40 m/s to 60 m/s over 5 s

The area is a trapezium.

$$\text{Distance}_2 = \frac{(v_1 + v_2)}{2} \times t_2$$

$$\text{Distance}_2 = \frac{(40 + 60)}{2} \times 5 = 250 \text{ m}$$

- ***Phase 3:*** Constant velocity 60 m/s for 8 s

$$\text{Distance}_3 = v_2 \times t_3 = 60 \times 8 = 480 \text{ m}$$

Total distance travelled

$$\text{Total distance} = \text{Distance}_1 + \text{Distance}_2 + \text{Distance}_3$$

$$= 320 + 250 + 480$$

$$= 1050 \text{ m}$$

Checkpoint 2.9.29 This question contains interactive elements.

Exercises

1. Given the table below

Table 2.9.30

Time	0	1	2	3	4	5	6	7
Velocity (m/s)	0	5	15	30	45	50	55	60

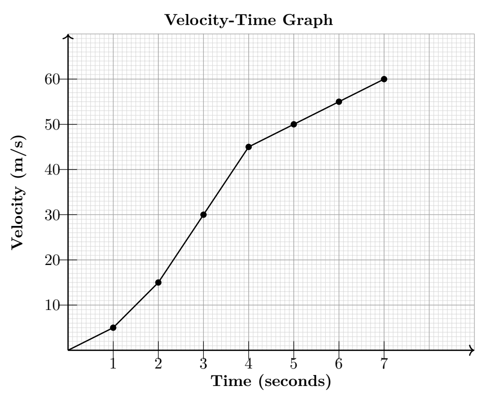
a) Draw the velocity time graph to represent the data

b) Use the graph to describe the motion of the vehicle. Is the velocity constant, increasing, or decreasing? Explain your answer.

c) Calculate the average velocity of the vehicle between t = 0 s and t = 3 s

Answer.

a) The velocity-time graph;



b) The motion of the vehicle is increasing as the velocity is increasing with time.

c) Average Velocity from t = 0 s to t = 3 s

$$\begin{aligned}\text{Average velocity} &= \frac{\text{Total distance}}{\text{Total time}}\\ &= \frac{(2.5 + 10 + 22.5)}{3}\\ &= 11.67 \text{ m/s}\end{aligned}$$

Therefore, the average velocity is 11.67 m/s

2. A motorcycle starts from rest and accelerates uniformly to a speed of 30 m/s in 8 seconds. It then continues at this speed for 12 seconds before decelerating uniformly to rest in 10 seconds.

a) Draw the velocity-time graph to represent its motion.

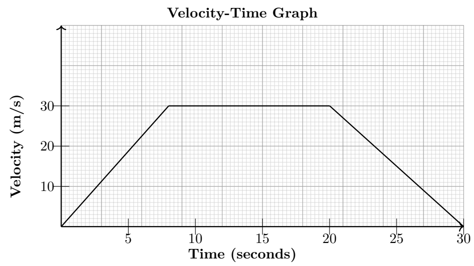
b) Find the total distance covered.

c) Determine the average velocity.

d) Calculate the acceleration and deceleration.

Answer.

a) The velocity-time graph;



b) The total distance is the area under the velocity-time graph, which consists of a trapezium.

$$= \frac{1}{2} \times 8 \times 30$$
$$= 120 \text{ m}$$
$$= 30 \times 12$$
$$= 360 \text{ m}$$
$$= \frac{1}{2} \times 10 \times 30$$
$$= 150 \text{ m}$$

Total distance $= 120 + 360 + 150 = 630$ m

c) Average Velocity $= \frac{630}{30} = 21$ m/s

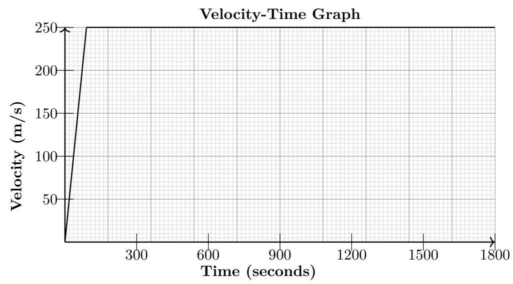
$$\text{Average velocity} = \frac{\text{Total distance}}{\text{Total time}}$$
$$= \frac{630}{30}$$
$$= 21 \text{ m/s}$$

d) The acceleration is calculated as: $a = \frac{\Delta v}{\Delta t} = \frac{30-0}{8} = 3.75 \text{ m/s}^2$.

The deceleration is calculated as: $a = \frac{\Delta v}{\Delta t} = \frac{0-30}{10} = -3 \text{ m/s}^2$.

3. After takeoff, an airplane reaches a cruising speed of 250 m/s and maintains it for 30 minutes. Draw a velocity-time graph representing the motion of the airplane from t = 0 to t = 1800 seconds.

Answer. The velocity-time graph;



4. A cyclist starts at 5 m/s and increases speed to 20 m/s in 15 seconds.

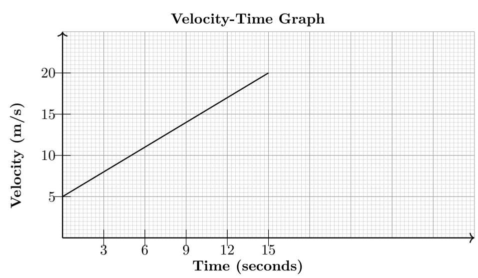
a) Draw the Velocity-Time Graph.

b) Calculate the acceleration of the cyclist.

c) What type of motion is represented by the velocity-time graph you drew?

Answer.

a) The velocity-time graph;



b) The acceleration is calculated as: $a = \frac{\Delta v}{\Delta t} = \frac{20-5}{15} = 1 \text{ m/s}^2$.

c) The motion represented by the velocity-time graph is uniformly accelerated motion since the cyclist's velocity increases at a constant rate over time.

2.9.6 Relative Speed

Relative speed is a simple concept that helps us understand how fast one object is moving compared to another object. It's important because sometimes objects are moving at different speeds, and we need to figure out how quickly they are closing the gap between them or moving away from each other.

Consider a case where two bodies moving in the same direction at different speeds. Their **relative speed** is the difference between the individual speeds.

But if they are moving toward each other, their relative speed is the sum of their speeds.

2.9.6.1 When Objects Move in the Same Direction

Activity 2.9.7 ***Work in Groups.***

In this activity you are required to follow each step as listed.

What you need:

- Toy car (that can be pulled by a string).
- String (long enough for the car to travel a sufficient distance).
- Stopwatch or timer
- Measuring tape or marked distance on the ground (e.g., 10 meters)
- Pen and paper (for recording results)
- Observers (to track the race and help evaluate results)
- Calculator (optional, for calculations)

Steps:

(i) Find a straight line or pathway that is clear and suitable for the race (a hallway, classroom floor, long corridor or outside on a field).

(ii) Measure out a distance for the race, such as 10 meters. You can use a measuring tape or use a marked area to set the starting and finishing points.

(iii) Mark the starting line and the finish line clearly, so both the toy car and the student know where to start and end.

(iv) Attach one end of the string securely to the toy car. The other end of the string should be held by a participant or fixed to an anchor (like a pole or sturdy object) so that when the student pulls the string, the car moves along the line and test the toy car to ensure that it moves easily along the straight path when the string is pulled.

(v) You will have two participants: one student (who will walk) and one toy car (which will be pulled by the string).

(vi) Assign at least one observer to track the race. The observer(s) will help measure the time it takes for each participant (the student and the toy car) to reach the finish line.

(vii) Place the student and the toy at the starting line. The student should be ready to walk at a constant speed.

(viii) On the count of ”Go!”, the student starts walking in a straight line towards the finish line. At the same time, the person holding the string should start pulling the toy car in the same direction (straight line) at a constant speed. The toy car should be pulled in a way that resembles a consistent movement, not too fast or too slow.

(ix) The observers should start the stopwatch as soon as both participants begin moving and stop it when either the student or the toy car reaches the finish line.

(x) Record the time taken by both the student and the toy car to reach the finish line. If you have more than one observer, make sure they agree on the recorded times.

(xi) After the race, you need to calculate the relative speed between the student and the toy car.

  - Let’s say the student took $T1$ seconds to cover the distance (10 meters), and the toy took $T2$ seconds to cover same distance

- The speed of the student is:

$$\frac{\text{Distance}}{\text{Time}} = \frac{10}{\text{T1 seconds}}$$

The speed of the toy car is:

$$\frac{\text{Distance}}{\text{Time}} = \frac{10}{\text{T2 seconds}}$$

- If the student and the toy car are moving in the same direction, the relative speed can be given as:
  Relative speed = Speed of Toy Car - Speed of Student

(xii) Compare and share your fndings with other groups of both the student and the toy car.

(xiii) Discuss;

(a). Who moved faster? Was the student walking faster or slower than the toy car?

(b). Why one object might have moved faster than the other. Did the toy car move faster because it was pulled, or was the student faster in their walking?

(c). The relative speed: How does the speed of each participant relate to the other? Did they move away from each other, or did they move closer together?

(xiv) Now try the activity in different conditions or compare vraious speed by repeating the race multiple times.

- Have the student walk faster or slower.
- Change the length of the race to see how it affects the results.
- Adjust how fast the toy car is pulled.

Example 2.9.31

A cyclist is riding at a speed of 12 *km/h*, and a motorcycle is moving at a speed of 20 *km/h* on the same road in the same direction. If the cyclist starts 100 meters ahead of the motorcycle, how long will it take for the motorcycle to overtake the cyclist?

Solution. Since the objects are moving in the same direction, we calculate the relative speed by subtracting the cyclist's speed from the motorcycle's speed:

$$RelativeSpeed = SpeedofMotorcycle\ -\ SpeedofCyclist$$

$$= 20\,km/h\ -\ 12\,km/h$$

$$= 8\,km/h$$

Now let's convert relative speed to meters per second, so that we can work with the distance in meters, we need to convert the relative speed from *km/h* to *m/s*.

$$Relative\,speed = 8\,km/h \times \frac{1000}{3600}$$

$$= 2.22\, m/s$$

The initial distance between the motorcycle and the cyclist is 100 meters. To calculate the time it will take for the motorcycle to overtake the cyclist,we have:

$$Time = \frac{Distance}{Relative\, Speed}$$

$$= \frac{100\, m}{2.22\, m/s}$$

$$\approx 45\, seconds$$

□

Example 2.9.32 A car travels at 60 *km/h*, and a truck travels at 45 *km/h* in the same direction on a straight road. If the car starts 150 meters behind the truck, how long will it take for the car to overtake the truck

Solution. The relative speed between the car and the truck is:

$$\text{Relative Speed} = \text{Speed of Car} - \text{Speed of Truck}$$

$$= 60\, km/h - 45\, km/h$$

$$= 15\, km/h$$

Convert Relative Speed to Meters per Second: Convert 15 km/h to m/s:

$$= 15\, km/h \times \frac{1000}{3600}$$

$$= 4.17\, m/s$$

The initial distance between the car and the truck is 150 meters. Use the formula to calculate the time:

$$\text{Times} = \frac{Distance}{Relative\, Speed}$$

$$= \frac{150m}{4.17\, m/s}$$

$$\approx 36\, seconds$$

□

Checkpoint 2.9.33 A cyclist starts from a point travelling at a constant speed of 12 km/h. Another cyclist starts from the same point, $\frac{2}{9}$ hours later, travelling in the same direction at a constant speed of 17 km/h.

How long after the second cyclist starts will the second cyclist catch up with the first?

Time = ______ hours

Answer. $\frac{8}{15}$

Solution. Worked Solution The first cyclist starts earlier, so the distance they cover before the second cyclist starts is given by its velocity multiplied by the time difference: $12 \times \frac{2}{9} = \frac{8}{3}$ km.

Since both cyclists travel in the same direction, the relative speed is: $17 - 12 = 5$ km/h.

Therefore, time taken to catch up is: $\frac{\frac{8}{3}}{5} = \frac{8}{15}$ hours.

Checkpoint 2.9.34 Two buses are traveling on the same road in the same direction. Bus A moves at 39 km/h, and Bus B moves at 32 km/h. At a certain moment, Bus A is 1054 m behind Bus B.

How long will it take for Bus A to overtake Bus B?

Time = ______ hours

Answer. $\frac{527}{3500}$

Solution. Worked solution Since both buses travel in the same direction, Bus A approaches Bus B at the ***relative speed***:

$39 - 32 = 7$ km/h

The initial distance between the buses is: $\frac{527}{500}$ km

The time needed to overtake is:

$$\begin{aligned} t =& \frac{\text{distance}}{\text{relative speed}} \\ =& \frac{\frac{527}{500}}{7} \\ =& \frac{527}{3500} \text{ hours} \end{aligned}$$

Exercises

1. A cyclist is riding at a speed of 18 *km/h*, and a motorcycle is traveling at 30 *km/h* on the same road. If the cyclist starts 200 meters ahead of the motorcycle, how long will it take for the motorcycle to overtake the cyclist?

Answer. The relative speed is $30 - 18 = 12$ *km/h*.
Converting to *m/s*, we get $12 \times \frac{1000}{3600} = \frac{10}{3}$ *m/s*.
The time taken to overtake is $\frac{200}{\frac{10}{3}} = 60$ seconds.

2. A toy car is being pulled along a straight path at a speed of 5 *m/s*, while a person walks at 2 *m/s* along the same path. If the toy car starts 15 meters ahead of the person, how much time will it take for the person to catch up with the toy car?

Answer. The relative speed is $5 - 2 = 3$ *m/s*.
The time taken to catch up is $\frac{15}{3} = 5$ seconds.

3. Two cyclists are riding along the same road. Cyclist A is traveling at 12 *km/h* and Cyclist B is traveling at 15 *km/h*. If Cyclist B starts 100 meters behind Cyclist A, how long will it take for Cyclist B to overtake Cyclist

A?

Answer. The relative speed is $15 - 12 = 3$ km/h.
Converting to m/s, we get $3 \times \frac{1000}{3600} = \frac{5}{6}$ m/s.
The time taken to overtake is $\frac{100}{\frac{5}{6}} = 120$ seconds.

4. A person walks at a speed of 1.5 m/s, and a dog runs at a speed of 3 m/s. If the dog starts 10 meters behind the person, how long will it take the dog to catch up with the person?

Answer. The relative speed is $3 - 1.5 = 1.5$ m/s.
The time taken to catch up is $\frac{10}{1.5} = 6.67$ seconds.

5. Two buses are traveling on the same road in the same direction. Bus A moves at 55 km/h, and Bus B moves at 45 km/h. If Bus A is 500 meters behind Bus B, how long will it take for Bus A to overtake Bus B?

Answer. The relative speed is $55 - 45 = 10$ km/h.
Converting to m/s, we get $10 \times \frac{1000}{3600} = \frac{25}{9}$ m/s.
The time taken to overtake is $\frac{500}{\frac{25}{9}} = 180$ seconds.

2.9.6.2 When Objects Move Toward Each Other (Opposite Directions).

Activity 2.9.8 Using **Activity 2.9.7.** above now try the race such that the racers move towards each other.

(a). Using the starting and finishing point.

- Put the Student on the marked end, and
- The Toy car on the other end.

(b). Measure time, record and calculate the speeds.

(c). If the student and the toy car are moving in the same direction, the relative speed is given as;

Relative speed = Speed of Toy Car + Speed of Student

(d). Discuss;

(i). Who moved faster? Was the student walking faster or slower than the toy car?

(ii). Why one object might have moved faster than the other. Did the toy car move faster because it was pulled, or was the student faster in their walking?

(iii). The relative speed: How does the speed of each participant relate to the other? Did they move away from each other, or did they move closer together?

(e). Now try the activity in different conditions or compare vraious speed by repeating the race multiple times.

- Have the student walk faster or slower.
- Change the length of the race to see how it affects the results.
- Adjust how fast the toy car is pulled.

Example 2.9.35 A train left town X at $10:00$ AM and traveled towards town Y at a speed of 90 km/h. A second train left town Y at $11:00$ AM

and traveled towards town X at 120 km/h. The distance between town X and town Y is 360 km.

(i). At what time will the two trains meet?

(ii). How far from town X will they meet?

Solution. The first train departs at $10:00\ AM$, and the second train departs at $11:00\ AM$. Therefore, the first train travels for 1 hour before the second train starts.

The distance traveled by the first train in 1 hour:

$$\text{Distance} = 90\,km/h \times \ \ hour$$

$$= 90\,km$$

Thus, by $11:00\ AM$, the remaining distance between the two trains is:

$$360\,km\ -\ 90\,km = 270\,km$$

The time it takes for the two trains to meet after 11:00 AM is given by the formula:

$$\text{Time} = \frac{\text{Distnace}}{\text{Relative Speed}}$$

$$= \frac{270\,km}{210\,km/h}$$

$$= 1.2857\,hours$$

This is approximately 1 hr and 17 minutes.

Therefore, the trains will meet at:

$$11:00\text{AM} + 1\text{ hour and 17 minutes} = 12:17\text{PM}$$

So the two trains meet at $12:17$PM

The distance traveled by the first train before the meeting is:

$$\text{Distance} = 90\,km/h \times 1.2857\,hours$$

$$= 115.714\,km$$

Thus, the meeting point is approximately $116\,km$ from town X. □

Example 2.9.36 Two cyclists start from the same point and travel in opposite directions. One cyclist rides at 20 km/h, and the other rides at 30 km/h. After 2 hours, they are 100 km apart.

(a). How long did it take for the cyclists to be 100 km apart?

(b). How far did each cyclist travel?

Solution. The cyclists are moving in opposite directions. The total distance between them after 2 hours is 100 *km*.

Since the cyclists are moving in opposite directions, their relative speed is the sum of their individual speeds:

$$\text{Relative speed} = 20\,km/h + 30\,km/h.$$

In 2 hours, the total distance traveled by both cyclists is:

$$\text{Total distance} = 50\,km/h \times 2\,hours$$

$$= 100\,km$$

So, the cyclists will be 100 km apart after 2 hours. This confirms the given information.

The first cyclist, riding at 20 km/h, travels:

$$\text{Distance covered by first cyclist} = 20\,km/h \times 2\,hours$$

$$= 40\,km.$$

The second cyclist, riding at 30 km/h, travels:

$$\text{Distance covered by second cyclist} = 30\,km/h \times 2\,hours$$

□

Checkpoint 2.9.37 A motorist left ***Nakuru*** for ***Nairobi***, a distance of 300 km, at 7:00 am and travelled at an average speed of 80 km/h.

Another motorist left ***Nairobi*** for ***Nakuru*** at 8:00 am and travelled at 90 km/h.

Find:

1. The time they met:8 : 00 am+______ hours
2. How far from Nairobi they met: ______ km

Answer 1. $\frac{22}{17}$

Answer 2. $\frac{1980}{17}$

Solution. Worked solution The first motorist starts earlier than the second. During the delay of 1 hours the first motorist travels:

80km/h × 1h = 80 km

When the second motorist starts, the remaining distance between them is:

300 km − 80 km = 220 km

Since the motorists are travelling towards each other, their Relative speed is:

$80 + 90 = 170$ km/h

The time taken from the second motorist's departure until they meet is:

$\frac{220}{170} = \frac{22}{17}$ hours

Therefore, the time they is:

Second motorist's start time + $\frac{22}{17}$ hours or first motorist's start time +$\frac{39}{17}$ hours

The distance from Nairobi is:

$90 \times \frac{22}{17} = \frac{1980}{17}$ km

Checkpoint 2.9.38 ***[[verbatim]]***

Alice and Bob start cycling toward each other from two towns 50 km apart. Bob cycles $\frac{17}{2}$ km/h faster than Alice. If they meet after 3 hours:

Find their speeds.

Bob's speed: ______ km/h

Alice's speed: ______ km/h

Answer 1. $\frac{27}{2}$

Answer 2. 5

Solution. Worked Solution Let Alice's speed = v km/h.

Then Bob's speed = $v + \frac{17}{2}$ km/h.

When cycling toward each other, their relative speed is the sum:

$v + (v + \frac{17}{2}) = 2v + \frac{17}{2}$ km/h

They cover 50 km in 3 hour, so:

$(2v + \frac{17}{2}) \times 3 = 50$

$2v + \frac{17}{2} = \frac{50}{3}$

$2v = \frac{50}{3} - \frac{17}{2}$

$v = \frac{\frac{50}{3} - \frac{17}{2}}{2} = 5$ km/h

Bob's speed $= v + \frac{17}{2} = 5 + \frac{17}{2} = \frac{27}{2}$ km/h

Therefore, their speeds are:

- Bob: $\frac{27}{2}$ km/h
- Alice: 5 km/h

Checkpoint 2.9.39 Two cyclists start 70 km apart and ride towards each other. Cyclist A rides at 17 km/h for the first 15 minutes, then continues at 14 km/h. Cyclist B rides at 12 km/h for the first 37 minutes, then continues at 10 km/h.

When do they meet?

______ hours

Answer. $\frac{1463}{480}$

Solution. Worked solution Cyclist A travels in the first 15 minutes:

$$17 \times \frac{15}{60} = \frac{17}{4} \text{ km.}$$

Cyclist B travels in the first 37 minutes:

$$12 \times \frac{37}{60} = \frac{37}{5} \text{ km.}$$

The remaining distance between them is:

$$70 - (\frac{17}{4} + \frac{37}{5}) = \frac{1167}{20} \text{ km.}$$

After both cyclists change speed, they approach each other at:

$$14 + 10 = 24 \text{ km/h.}$$

Cyclist B finishes the first stage after 37 minutes.

Therefore, the time until both cyclists have changed speed is:

$$\frac{37}{60} \text{ hours.}$$

After that moment, the cyclists meet after an additional:

T=D/S = $\frac{389}{160}$ hours.

Hence, the total time until they meet is:

$$\frac{37}{60} + \frac{389}{160} = \frac{1463}{480} \text{ hours.}$$

Therefore, they meet at: $\frac{1463}{480}$ hours

Exercises

1. A motorist left Nakuru for Nairobi, a distance of 240 km, at 8.00 am. and travelled at an average speed of 90 km/h. Another motorist left Nairobi for Nakuru at 8.30 *am.* and travelled at 100 *km/h.* Find:

(i). The time they met.

(ii). How far from Nairobi they met.

Answer. The time taken for the two motorists to meet after 8.30 am is given by:

$$\begin{aligned} &= \frac{195\,km}{190\,\text{km/h}} \\ &= 1.0263\,\text{hours} \end{aligned}$$

Therefore, the two motorists meet at:

$$8:30\,\text{am} + 1 \text{ hour and } 1.58 \text{ minutes} = 9:31.58\,\text{am}$$

The distance traveled by the second motorist before the meeting is:

$$\begin{aligned} &= 100\,km/h \times 1.0263\,\text{hours} \\ &= 102.63\,\text{km} \end{aligned}$$

Thus, they met approximately 102.63 km from Nairobi.

2. A train travels from Mombasa to Nairobi, a distance of 500 *km*, at a speed of 90 *km/h.* If a second train starts 1 hour later from Mombasa and travels at a speed of 120 km/h, after how much time will the second train overtake the first one?

Answer. The time taken for the second train to overtake the first train is given by:

$$\begin{aligned} &= \frac{90\,km}{30\,\text{km/h}} \\ &= 3\,\text{hours} \end{aligned}$$

3. A matatu left town A at 7 *am* and travelled towards a town B at an average speed of 60 *km/h.* A second matatu left town B at 8 *am* and travelled towards town A at 60 *km/h.* If the distance between the two towns is 400 *km*, find:

a. The time at which the two matatus met.

b. The distance of the meeting point from town A.

Answer.

(a) The time taken for the two matatus to meet after 8 am is given by:

$$= \frac{340\, km}{120\, \text{km/h}}$$
$$= 2.8333\, \text{hours}$$

Therefore, the two matatus meet at: $8 : 00\, \text{am} + 2\text{ hours and } 50\text{ minutes} = 10 : 50\, \text{am}$

(b) The distance traveled by the first matatu before the meeting is:

$$= 60\, km/h \times 2.8333\, hours$$
$$= 170\, km$$

4. A cyclist is riding towards a motorcyclist on a straight road. The cyclist travels at 15 *km/h* and the motorcyclist at 45 *km/h*. If they are initially 100 meters apart, how long will it take for the motorcyclist to overtake the cyclist?

Answer. The relative speed is $15 + 45 = 60$ *km/h*.
Converting to m/s, we get $60 \times \frac{1000}{3600} = 16.67$ *m/s*.
The time taken to overtake is $\frac{100}{16.67} = 6$ seconds.

5. Two cars, A and B, are traveling on parallel roads. Car A moves at a speed of 50 *km/h* and car B moves at 70 *km/h*. If car B is 150 meters behind car A, how long will it take for car B to overtake car A?

Answer. The relative speed is $70 - 50 = 20$ *km/h*.
Converting to m/s, we get $20 \times \frac{1000}{3600} = 5.56$ *m/s*.
The time taken to overtake is $\frac{150}{5.56} = 27$ seconds.

Chapter 3

Statistics and Probability

3.1 Statistics 1

Statistics is the branch of mathematics that deals with collecting, organizing, analyzing and interpreting data in order to make decisions or draw conclusions.

In our daily lives, we are constantly surrounded by data from school exam results, sports scores, population figures, business sales, weather reports, to health records.

Understanding statistics helps us make sense of this information and use it to solve real life problems.

3.1.1 Collection of Data

Collection of data is the process of gathering information or facts for analysis. In statistics, data can be collected to answer questions, test ideas or solve problems in real-life situations.

3.1.1.1 Sources of Data

Activity 3.1.1 Work in groups

A teacher assigned students a research topic on the effects of social media on teenage mental health in Kenya.

1. Suggest one potential primary data collection method that could be used.
2. Suggest one potential secondary data source that could be used.
3. What is the difference between primary and secondary data?
4. Why is secondary data important for research and decision-making?
5. Discuss and share with other groups.

Key Takeaway 3.1.1 Data can be obtained from primary sources, which are original first-hand data collected directly for a specific purpose (e.g., surveys, interviews),

secondary sources, which are previously collected data used for a different purpose (e.g., government reports, research articles), and tertiary sources, which summarize and compile information from primary and secondary sources (e.g., dictionaries, textbooks, encyclopedias).

Example 3.1.2 Classify the Following as Primary or Secondary Data Sources:

1. A student conducts a survey to find out the favorite sports of their classmates.
2. A teacher uses last year's national exam results to analyze student performance trends.
3. A researcher reads a government report on the most common diseases in Kenya.
4. A doctor observes a patient's symptoms and records them for a medical study.
5. A scientist conducts an experiment to test the growth rate of plants under different conditions.

Solution.

1. Primary because the student is collecting first-hand data directly from people.
2. Secondary because the exam results were already collected and recorded by an external body.
3. Secondary because The report was collected and published by someone else for a different purpose.
4. Primary because The doctor is directly collecting new data from real-life observation.
5. Primary because The scientist is generating new data through an experiment.

□

Example 3.1.3 A community group wants to understand the needs of youth in their area.

a) What primary data collection methods could the group use?

b) What problems might they have when trying to find this information?

c) How can they use the information they find to help the community?

d) What secondary data sources could be useful to this community group?

Solution.

a) Focus groups, surveys, interviews.

b)
- Overcoming language barriers.
- Getting honest responses.

c)
- They can start new programs for young people.
- They can tell leaders what young people need.

d)
- Academic studies on youth issues.
- Government reports on youth.

□

Exercises

1. Grade 10 students from Korinyang Primary School went to Lake Nakuru National Park and counted the flamingos they saw for their Biology project.

a) Is this a primary or secondary data source?

b) Give reason.

2. In a market, a shop owner watches what customers buy most often and looks for trends based on what they see in the store.
Which sources of data can the shop owner use to collect the data?

3. Define the following terms.

a) Data Source

b) Raw Data

c) Tertiary Sources

d) Primary sources

4. What are the advantages of using secondary data over primary data in some research situations?

Checkpoint 3.1.4 Classify each of the following data collection activities as either ***Primary Data*** or ***Secondary Data***:***(a)*** You interview 20 students to ask them about their favorite lunch meal.[Primary Data | Secondary Data] ***(b)*** You search the internet for the current exchange rate of the Dollar.[Secondary Data | Primary Data] ***(c)*** You hand out a questionnaire to your neighbors about water usage.[Secondary Data | Primary Data] ***(d)*** You use a graph from a textbook to analyze rainfall patterns from 1990.[Primary Data | Secondary Data]

Answer 1. Primary Data

Answer 2. Secondary Data

Answer 3. Primary Data

Answer 4. Secondary Data

Solution. Worked solution

- ***Primary Data:*** This is data that you collect *yourself* for a specific purpose. It is "fresh" and original. (Examples: Counting cars, interviewing people, measuring height).

- ***Secondary Data:*** This is data that *someone else* has already collected. You are just using it. (Examples: Census reports, newspaper articles, textbooks, internet searches).

Checkpoint 3.1.5 Select the correct definition for each of the following statistical terms from the dropdown menus:***(a) Data Source***[The final report generated after data analysis is complete. | The mathematical formula used to calculate the mean. | The specific origin, location, or medium from where statistical facts are obtained. | The software tools used to analyze numerical information.] ***(b) Raw Data***[The average value calculated from a set of numbers. | Information that has been sorted alphabetically. | Data recorded in its original form as collected, without any organization or processing. | Data that has been rounded off to the nearest decimal.] ***(c) Tertiary Sources***[

The raw responses collected immediately from a survey. | Detailed analysis and interpretation of data found in academic journals. | Sources that compile, index, or summarize primary and secondary sources (e.g., encyclopedias). | First-hand accounts and original data collected directly by a researcher.] ***(d) Primary Sources***[Data that has already been manipulated and published by another author. | Information obtained from existing records like government censuses or books. | Original data or first-hand information collected directly by the researcher for a specific purpose. | Abstracts and indexes used to locate other research materials.]

Answer 1. The specific origin, location, or medium from where statistical facts are obtained.

Answer 2. Data recorded in its original form as collected, without any organization or processing.

Answer 3. Sources that compile, index, or summarize primary and secondary sources (e.g., encyclopedias).

Answer 4. Original data or first-hand information collected directly by the researcher for a specific purpose.

Solution. Worked solution

- ***Data Source:*** The origin or specific location from where statistical facts are obtained.
- ***Raw Data:*** Data recorded in its original form as collected (ungrouped), without any organization or processing.
- ***Tertiary Sources:*** These are sources that facilitate the identification and location of primary and secondary sources. Examples include bibliographies, indexes, and encyclopedias.
- ***Primary Sources:*** Original materials or first-hand information collected directly by the researcher for a specific purpose (e.g., surveys, interviews).

3.1.1.2 Methods of Data Collection

Activity 3.1.2 Work in groups

1. The following data represents the number of hours people spend watching television per week.
   - $0 - 5$ hours: 20 people
   - $6 - 10$ hours: 35 people
   - $11 - 15$ hours: 25 people
   - $16 - 20$ hours: 10 people
2. Create a table to organize this data.
3. What type of data collection method was most likely used to gather this data?
4. What are some other questions that could be asked to further explore this topic?
5. Discuss and share with other groups.

Key Takeaway 3.1.6

- Methods of collecting data help us gather information. We can collect data directly through surveys, interviews, or experiments, or indirectly from books, reports, and online sources.

Example 3.1.7 A student wants to find out the most popular extracurricular activities among their classmates.

a) What data collection method would be most appropriate?

b) What are two examples of specific questions they could ask?

Solution.

a) Survey/Questionnaire.

b) Questions for students to ask are;

   - What extracurricular activities do you participate in?
   - How often do you participate in these activities?

□

Example 3.1.8 A local bakery, conducted a study asking customers about their favorite types of pastries. The results are:

- Cakes: 45%
- Cookies: 30%
- Breads: 20%
- Doughnuts: 5%

a) What type of data collection method was most likely used to gather this data?

b) What is the most popular type of pastry among customers?

c) What percentage of customers prefer either Cakes or Cookies?

d) What type of pastry is the least popular among customers?

Solution.

a) A survey.

b) Cakes

c) 45% + 30%= 75%

   Therefore, the percentage of customers who prefer either Cakes or Cookies are 75%

d) Doughnuts

□

Exercises

1. Which data collection method would be most suitable for finding out the most common types of litter found in your school compound?

a) Interviews

b) Surveys sent to parents

c) Observations

d) Analyzing government reports

2. Describe one ethical consideration you should keep in mind when conducting interviews with community members.

3. Why is it important to keep accurate records when conducting observations?

4. Give one example of a situation where you would use secondary data collection in a Geography class.

Checkpoint 3.1.9 Consider the following research scenario:

The government requires every single citizen to provide details on their age, occupation, and housing status to assist in national planning.

Identify the specific method of data collection being used in this situation.

(1) Case Study

(2) Focus Group

(3) Census

(4) Questionnaire

Answer. (3)

Solution. ***Correct Answer: Census***

Reasoning:

- ***Observation:*** Involves watching subjects in their natural environment without interaction (e.g., the wildlife ranger).
- ***Focus Group:*** A guided discussion with a small group to get qualitative opinions (e.g., the teenagers discussing phones).
- ***Census:*** A complete count of the entire population (e.g., the government survey).
- ***Experimentation:*** Manipulating variables (fertilizer) to measure the effect (maize growth).
- ***Questionnaire:*** Using standard forms/emails to get feedback from a large number of people (e.g., the student email).
- ***Secondary Data:*** Analyzing data that was already collected by someone else (e.g., the NBS reports).

Checkpoint 3.1.10 Read the following research scenario:

You gather detailed opinions and feelings from a small group of 10 people regarding a new product.

Which method of data collection is most appropriate for this scenario?

(1) Questionnaire

(2) Secondary Data Collection

(3) Telephone Survey

(4) Focus Group

Answer. (4)

Solution. ***Correct Answer: Focus Group***

Each method serves a specific purpose:

- ***Observation:*** Studying behavior in natural settings.
- ***Focus Group:*** Gathering deep qualitative insights from small groups.
- ***Census:*** Collecting data from the entire population.
- ***Experimentation:*** Establishing cause-and-effect (control vs treatment).
- ***Questionnaire:*** Gathering standardized data from large groups efficiently.
- ***Secondary Data:*** Using existing records or data collected by others.

3.1.2 Representing Data using a Frequency Distribution Table

Activity 3.1.3 Work in groups

1. Below are the weekly pocket expenses (in Ksh) of a randomly selected group of 25 students.

   $120, 150, 180, 200, 220, 250, 270, 290, 300, 320, 350, 370, 390, 400,$

   $420, 450, 470, 480, 490, 500, 340, 230, 280, 410, 330$

2. Draw a grouped frequency distribution table with 5 classes to represent the data?

3. Identify the number with the highest frequency.

4. Compare your answer with others in class.

Key Takeaway 3.1.11 A frequency distrribution table is a table that shows an event and how many times it happens.

There are two types:

- **Ungrouped Frequency Distribution**: for small datasets with individual values.
- **Grouped Frequency Distribution**: for large datasets where values are grouped into intervals.

  Steps to construct a grouped frequency distribution table

  1. Determine the Range of Data

  $$\textbf{Range} = \textbf{Maximum Value} - \textbf{Minimum Value}$$

  2. Decide the Number of Classes (Groups)

3. Calculate the Class Width

$$\textbf{Class width} = \frac{\textbf{Range}}{\textbf{Number of classes}}$$

4. Establish Class Boundaries
   - Begin with the Lowest Value: Use the smallest data value as the lower limit of the first class.
   - Determine the Upper Limit: Add the class width to establish the upper limit of the first class and the lower limit of the next class.
   - Repeat the Process: Continue this pattern until all class intervals are created.
5. Tally the Frequencies
   - Count how many data points fall within each class interval and record the frequency.
6. Complete the Table
   - Class Interval
   - Tally Marks
   - Frequency(f)

Example 3.1.12 The following data represents test scores of 20 students in a grade 10 class.

$45, 50, 55, 50, 60, 70, 75, 80, 70, 55, 60, 65, 50, 55, 45, 60, 75, 80, 70, 50$

Prepare ungrouped frequency distribution table for the dataset.

Solution.

Table 3.1.13

Test scores	**Tally**	**Frequency**
45	//	2
50	////	4
55	///	3
60	////	4
65	/	1
70	///	3
75	//	2
80	//	2

□

Example 3.1.14 The number of customers visiting a supermarket over 30 days were recorded as follows:

$135, 125, 140, 160, 145, 120, 150, 140, 130, 125, 135, 155, 140, 135, 130,$
$155, 150, 160, 145, 140, 120, 145, 135, 140, 150, 130, 150, 125, 145, 120$

Draw a grouped frequency distribution table with 5 classes to represent the data?

Solution. To prepare frequency table for the grouped data above, we need to first find the range for the data.

$$\textbf{Range} = \textbf{Maximum Value} - \textbf{Minimum Value}$$

Maximum value = 160

Minimum value = 120

$$
\begin{aligned}
\textbf{Range} =& 160 - 120 \\
& 40
\end{aligned}
$$

Range is 40
Next, Determine Class Width

$$
\begin{aligned}
\textbf{Class width} =& \frac{\textbf{Range}}{\textbf{Number of classes}} \\
& \frac{40}{5} = 8
\end{aligned}
$$

Class widths are 8
Create Class Intervals
Starting from 120, we create intervals of width 8:
120 − 127
128 − 135
136 − 143
144 − 151
152 − 160
Tally the Data
We count how many values fall into each interval.
120 − 127: 120, 120, 125, 125, 125
128 − 135: 130, 130, 130, 135, 135, 135, 135
136 − 143: 140, 140, 140, 140, 140, 140
144 − 151: 145, 145, 145, 145, 150, 150, 150, 150
152 − 160: 155, 155, 160, 160
Then construct the frequency Table

Table 3.1.15

Test scores	**Tally**	**Frequency**
120 − 127	~~////~~	5
128 − 135	~~////~~ //	7
136 − 143	~~////~~ /	6
144 − 151	~~////~~ ///	8
152 − 160	////	4

□

Exercises

1. Twenty five students in Grade 10 recorded their time travel to school in minutes as follows:
15, 8, 22, 30, 12, 25, 18, 10, 35, 20, 5, 28, 15, 40, 17, 23, 12, 32, 7, 19, 27, 14, 21, 9, 33
Draw a frequency distribution table to represent the data.

2. Thirty customers at a service center recorded their wait times in minutes as follows:
5, 12, 8, 15, 3, 10, 18, 6, 13, 9, 2, 16, 11, 7, 19, 14, 4, 8, 12, 20, 5, 17, 9, 13, 6, 11, 3, 15, 8, 14.
Prepare a frequency distribution table for the set of data.

3. The costs (in Ksh.) of manufacturing equipment across different factories were recorded as follows:
1250, 1425, 1580, 1720, 1850, 1975, 2100, 2235, 2370, 2480, 2610, 4310, 4425,

2750, 2880, 3025, 3150, 3280, 3410, 3525, 3640, 3750, 3870, 3975, 4080, 4195, 4550, 4680, 4820, 4950, 1380, 1625, 1890, 2340, 2570, 2780, 3120, 3390, 3610, 4520, 4750, 3150, 2840, 1950, 2640, 3470, 4180, 3840, 4030, 4270.

Prepare a frequency distribution table for the grouped data.

4. The annual rainfall (in mm) recorded in a region was as follows:

625, 645, 670, 695, 720, 745, 770, 790, 810, 835, 860, 880, 905, 1000, 1025, 1050, 1075, 1100, 1125, 1150, 930, 950, 975, 1180.

Construct a grouped frequency distribution table for the data.

Checkpoint 3.1.16 Thirty customers at a service center recorded their wait times in minutes as follows:

22, 3, 11, 11, 10, 24, 19, 6, 22, 24, 10, 10, 22, 3, 10, 6, 3, 22, 22, 11, 22, 10, 10, 24, 24, 6, 6, 19, 11, 11

Prepare a frequency distribution table for this set of data. The wait times observed have been listed in the first column for you. Count how many times each wait time appears in the list above.

Wait Time (Minutes)	Frequency
3	______
6	______
10	______
11	______
19	______
22	______
24	______

Answer 1. 3

Answer 2. 4

Answer 3. 6

Answer 4. 5

Answer 5. 2

Answer 6. 6

Answer 7. 4

Solution. Worked Solution To solve this, you count how many times each specific number appears in the list.

- Wait time 3 appears 3 times.
- Wait time 6 appears 4 times.
- Wait time 10 appears 6 times.
- Wait time 11 appears 5 times.
- Wait time 19 appears 2 times.
- Wait time 22 appears 6 times.
- Wait time 24 appears 4 times.

Total: $3 + 4 + 6 + 5 + 2 + 6 + 4 = 30$.

Checkpoint 3.1.17 The following data represents the exam scores of 36 students in a mathematics class:

61, 89, 82, 53, 60, 55, 88, 60, 76, 60, 56, 77, 44, 54, 47, 79, 88, 68, 47, 40, 74, 70, 49, 65, 53, 74, 68, 63, 83, 55, 85, 45, 70, 84, 42, 53

Construct a ***grouped frequency distribution table*** for this data using the class intervals provided below.

Score Interval	Frequency
40 – 49	______
50 – 59	______
60 – 69	______
70 – 79	______
80 – 89	______
Total	______

Answer 1. 7

Answer 2. 7

Answer 3. 8

Answer 4. 7

Answer 5. 7

Answer 6. 36

Solution. Worked Solution To create the grouped frequency table, you tally the scores that fall into each range:

- 40 – 49***:*** Counting scores in the 40s gives 7.
- 50 – 59***:*** Counting scores in the 50s gives 7.
- 60 – 69***:*** Counting scores in the 60s gives 8.
- 70 – 79***:*** Counting scores in the 70s gives 7.
- 80 – 89***:*** Counting scores in the 80s gives 7.

Total Check: $7 + 7 + 8 + 7 + 7 = 36$.

3.1.3 Measures of Central Tendency

Measures of central tendency are statistical values that describe the center or average of a set of data.

They help us identify a single number that represents the entire distribution of data, giving us an idea of what is "typical" or "common."

The three main measures are Mean, Median and Mode.

Together, these measures give us different perspectives of the center of the data.

3.1.3.1 Ungrouped Data for Measures of Central Tendency

Ungrouped data is the raw form of data, where individual observations are listed as they are collected, without being organized into groups or classes

It is simply a list of numbers, facts, or values recorded in the order they are obtained

Activity 3.1.4 Work in groups

1. Consider the following data set:

   $32, 33, 35, 36, 38, 40, 41, 42, 44, 45, 47, 48, 50, 52, 54, 55, 56, 57,$
   $58, 60, 62, 63, 65, 66, 68, 70, 72, 74, 75, 78, 80, 82, 85$

2. Construct a frequency distribution table for the data.

3. Find the Mean, Mode and Median for the data.
4. Discuss with other groups

Key Takeaway 3.1.18 There are three main measures of central tendency:

1. **Mean**

   Mean is the sum of all values divided by the total number of values. It is also known as Arithmetic Average.

   $$\textbf{Mean} = \frac{\textbf{X}}{\textbf{N}}$$

   Where **X** represents the values in the dataset and **N** is the total number of values.

   Frequency distribution table can be used to find the mean for the ungrouped data. Using the formula below:

   $$\overline{\textbf{x}} = \frac{\textbf{fx}}{\textbf{f}}$$

   Where;

   - **x** represents the values in the dataset
   - $\overline{\textbf{x}}$ is the mean
   - **fx** is the sum of products of **x** and **f**
   - **f** is the Sum of frequencies

2. **Median**

   Median is the middle value when the data is arranged in ascending or descending order. If the dataset has an even number of values, the median is the average of the two middle values.

   To find the median for even numbers, the formula is;

   $\textbf{Median} = \frac{\left(\frac{n}{2}\right)^{\text{th}} + \left(\frac{n}{2}+1\right)^{\text{th}}}{2}$

3. **Mode**

   Mode is the most frequently occurring value in the dataset. A dataset can have:

   - No mode: if no value repeats.
   - One mode (Unimodal): if one value appears most frequently.
   - Two modes (Bimodal): if two values appear equally most frequently.
   - Multimodal: if more than two values appear frequently.

Example 3.1.19 In a class of 30 students, the test scores of students are:
$45, 67, 89, 56, 45, 78, 90, 67, 81, 73, 55, 62, 77, 84, 91,$
$69, 58, 72, 88, 95, 60, 75, 45, 67, 80, 92, 87, 79, 68, 55$
Find the:

a) Mean

b) Median

c) Mode

Solution.

a) The formula for the mean is:

$$\textbf{Mean} = \frac{\mathbf{X}}{\mathbf{N}}$$

Where;

- X is the sum of all values
- N is the total number of values

$X = 45 + 67 + 89 + 56 + 45 + 78 + 90 + 67 + 81 + 73 + 55 + 62$
$+77 + 84 + 91 + 69 + 58 + 72 + 88 + 95 + 60 + 75 + 45 + 67$
$+80 + 92 + 87 + 79 + 68 + 55$
$= 2096$

$$= \frac{2096}{30}$$
$$= 69.87$$

Therefore, the mean is 69.87

b) Median

The median is the middle value when data is arranged in ascending or descending order.

$45, 45, 45, 55, 55, 56, 58, 60, 62, 67, 67, 67, 68, 69, 72,$

$73, 75, 77, 78, 79, 80, 81, 84, 87, 88, 89, 90, 91, 92, 95$

Since there are 30 values (even number), the median is;

$\textbf{Median} = \frac{\left(\frac{n}{2}\right)^{\text{th}} + \left(\frac{n}{2}+1\right)^{\text{th}}}{2}$

$= \frac{\left(\frac{30}{2}\right)^{\text{th}} + \left(\frac{30}{2}+1\right)^{\text{th}}}{2}$

$15\textbf{th} = 72$

$16\textbf{th} = 73$

$$\textbf{Median} = \frac{72 + 73}{2}$$
$$= \frac{145}{2}$$
$$= 72.5$$

Therefore the median is 72.5

c) Mode

The mode is the most frequently occurring values.

From the sorted data:

- 45 appears 3 times
- 67 appears 3 times

Since 45 and 67 appear most frequently, this dataset is bimodal with modes: 45 and 67.

□

Example 3.1.20 The frequency distribution table below shows marks of 20 students in a Grade 10 class.

Table 3.1.21

Marks (x)	**Frequency (f)**	**fx**
2	3	6
4	2	8
6	4	24
7	3	21
9	5	45
11	2	22
12	1	12
$x = 51$	$f = 20$	$fx = 138$

a) Find the mean

b) find the mean

Solution.

a) To find the mean

$$\begin{aligned}\overline{\mathbf{x}} =& \frac{\mathbf{fx}}{\mathbf{f}} \\ =& \frac{138}{20} \\ =& 6.9\end{aligned}$$

Therefore, the mean is 6.9

b) The mode is the mark with the highest frequency. In this case, the highest frequency is 5, which corresponds to 9.

Therefore, Mode is 9

□

Exercises

1. The number of books borrowed by students from a school library in a week is as follows:

$3, 5, 2, 4, 6, 3, 5, 7, 4, 3, 6, 2, 4, 5, 3, 6$

a) Find the Mean (Average) number of books borrowed.

b) Find the Median number of books borrowed.

c) Find the Mode of the books borrowed.

2. The following frequency distribution table represents volume of water (in liters) contained in different bottles:

Table 3.1.22

Volume of water (liters)	Number of bottles
34.5	3
35.8	4
37.2	2
39.0	3
40.4	3

a) Find the mean volume of water in the bottles.

b) Determine the mode of the data.

c) Find the median volume.

3. A company records the monthly salaries (in KES) of its employees.

Table 3.1.23

Salary (KES)	Frequency
25,000	8
30,000	15
35,000	12
40,000	10
50,000	5

Calculate the mean, median and mode.

4. A factory records the number of products manufactured in a week:

$150, 130, 220, 135, 180, 140, 125, 250, 145, 230, 200, 205, 145, 190, 155,$
$210, 225, 240, 135, 165, 245, 170, 175, 185, 130, 190, 195, 160, 170, 150,$
$160, 120, 200, 210, 220, 235, 180, 230, 240, 215$

a) Make a frequency distribution table for the set of data.

b) Calculate the mean, mode and median.

Checkpoint 3.1.24 The number of cement bags manufactured daily at the ***Simba Cement Factory*** over a period of 30 days is recorded below:[450, 505, 505, 515, 495, 490, 465, 505, 520, 475, 485, 450, 525, 475, 535, 490, 550, 515, 455, 540, 520, 520, 505,
Analyze this ungrouped data to determine the following measures of central tendency:

a) Mean: Calculate the mean daily production. Mean = ______ bags

b) Median: Determine the median number of bags produced. Median = ______ bags

c) Mode: Identify the mode of the dataset. *(Note: If there is more than one mode, enter them as a set, e.g.,* `{450, 460}`*).* Mode = ______

Answer 1. 499.0

Answer 2. 505.0

Answer 3. 505

Solution. ***a) Mean***

The mean is the sum of all observations divided by the total number of observations (N).$\bar{x} = \frac{\sum x}{N}$ $\bar{x} = \frac{14970}{30}$ $\bar{x} \approx 499$ bags ***b) Median***

To find the median, we first arrange the data in ascending order:[450, 450, 455, 460, 460, 465, 475, 475, 480, 485, 490, 490, 495, 495, 505, 505, 505, 505, 515, 515, 520, 520, 520, 5
Since there are $N = 30$ observations (an even number), the median is the average of the two middle values (the 15^{th} and 16^{th} terms).

- 15^{th} value: 505
- 16^{th} value: 505

Median $= \frac{505+505}{2} = 505.0$ bags ***c) Mode***

The mode is the value that appears most frequently in the dataset. To see this clearly, we can tally the frequency of each production amount:

Production (Bags)	Frequency
450	2
455	1
460	2
465	1
475	2
480	1
485	1
490	2
495	2
505	4
515	2
520	3
525	2
530	2
535	1
540	1
550	1

From the table, the highest frequency is 4.

The value(s) with this frequency: 505.

Checkpoint 3.1.25 The monthly salaries (in KES) of ***55 employees*** at ***Kabras Sugar Company*** are listed below:[28000, 38000, 45000, 30000, 50000, 38000, 40000, 55000, 50000, 25000, 50000, 55000, 42000, 45000, 25000, 4
Analyze this ungrouped data to determine the measures of central tendency.

a) Mean Salary: Calculate the mean monthly salary. Mean = KES _____

b) Median Salary: Determine the median monthly salary. Median = KES ______

c) Mode Salary: Identify the mode of the dataset. *(Note: If there is more than one mode, enter your answer as a set, e.g.,* `{30000, 35000}`*).* Mode = KES ______

Answer 1. 41527.2727273

Answer 2. 40000

Answer 3. 55000

Solution. ***a) Mean Salary***

The mean is the sum of all salaries divided by the total number of employees $(N = 55)$.$\bar{x} = \frac{\sum x}{N}$ $\bar{x} = \frac{2284000}{55}$ $\bar{x} \approx \mathbf{41527.27}$ KES ***b) Median Salary***

To find the median, we first arrange the salaries in ascending order:

$[25000, 25000, 25000, 25000, 25000, 28000, 28000, 28000, 28000, 30000, 30000, 30000, 30000, 30000, 30000, 3200$

Since $N = 55$ is an ***odd number***, the median is exactly the middle value.

Position of Median: $\frac{N+1}{2} = \frac{55+1}{2} = 28^{\text{th}}$ value.

Counting to the 28^{th} value in the sorted list, we find:Median = **40000** KES

c) Mode Salary

The mode is the salary that appears most frequently. We can verify this by organizing the data into a frequency table:

Salary (KES)	Frequency
25000	5
28000	4
30000	6
32000	2
35000	6
38000	3
40000	3
42000	2
45000	6
50000	3
55000	9
60000	6

The highest frequency is 9.

The salary with this frequency is: 55000.

3.1.3.2 Grouped Data for Measures of Central Tendency

Grouped data is data that has been organized into classes or intervals together with their frequencies.

Instead of listing every single value (as in ungrouped data), the observations are arranged into groups (class intervals) to make large data sets easier to understand, analyze, and interpret.

Activity 3.1.5 Work in groups

The amount of pocket money, in shillings, that parents give to students per week.

Table 3.1.26

Pocket Money (Ksh)	100 − 199	200 − 299	300 − 399	400 − 499	500 − 599
Number of students	8	15	22	20	10

1. What is the modal class of pocket money given to students?
2. Calculate the mean of pocket money given to students per week.
3. Find the median amount of pocket money from the given data.
4. Discuss with other groups.

Key Takeaway 3.1.27

1. **Mean**

   Frequency distribution table can be used to find the mean for the grouped data. Using the formula below:

$$\overline{\mathbf{x}} = \frac{\mathbf{fx}}{\mathbf{f}}$$

Where;

- $\mathbf{x}$ is the midpoint
- $\overline{\mathbf{x}}$ is the mean
- $\mathbf{fx}$ is the sum of products of $\mathbf{x}$ and $\mathbf{f}$
- $\mathbf{f}$ is the Sum of frequencies

Midpoint is the average of the lower and upper boundaries of a class interval. It represents the central value of each class.

$$\textbf{Midpoint} = \frac{\textbf{Lower boundary + Upper boundary}}{2}$$

2. **Median**

   For grouped frequency data, we use interpolation to estimate the median using the following formula;

$$\textbf{Median} = \mathbf{L} + \left(\frac{\frac{n}{2} - \mathbf{CF}}{\mathbf{F}}\right) \times \mathbf{C}$$

   Where;

   - $\mathbf{L}$ = Lower boundary of the median class.
   - $\mathbf{n}$ = sum of all frequencies.
   - $\mathbf{CF}$ = Cumulative frequency before the median class.
   - $\mathbf{F}$ = Frequency of the median class.
   - $\mathbf{C}$ = class width

3. **Mode**

   The Modal class is the class with highest frequency.

Example 3.1.28 A company records the monthly salaries (in KES) of 50 employees in a frequency distribution table below:

Table 3.1.29

Salary Range(KES)	**Number of Employees**
$20,000 - 29,999$	3
$30,000 - 39,999$	5
$40,000 - 49,999$	7
$50,000 - 59,999$	10
$60,000 - 69,999$	9
$70,000 - 79,999$	6
$80,000 - 89,999$	5
$90,000 - 99,999$	3
$100,000 - 109,999$	2

a) Find the mean and mode of the data.

b) Find the modal class of the data.

Solution.

a) To calculate the mean of the grouped data we need to find;

- **fx** which is the sum of products of **x** and **f**
- **f** is the sum of frequencies

Table 3.1.30

Salary Range(KES)	**Midpoint(x)**	**Number of Employees(f)**	**xf**
$20,000 - 29,999$	$25,000$	3	$25,000 \times 3 = 75,000$
$30,000 - 39,999$	$35,000$	5	$35,000 \times 5 = 175,000$
$40,000 - 49,999$	$45,000$	7	$45,000 \times 7 = 315,000$
$50,000 - 59,999$	$55,000$	10	$55,000 \times 10 = 550,000$
$60,000 - 69,999$	$65,000$	9	$65,000 \times 9 = 585,000$
$70,000 - 79,999$	$75,000$	6	$75,000 \times 6 = 450,000$
$80,000 - 89,999$	$85,000$	5	$85,000 \times 5 = 425,000$
$90,000 - 99,999$	$95,000$	3	$95,000 \times 3 = 285,000$
$100,000 - 109,999$	$105,000$	2	$105,000 \times 2 = 210,000$
Total ( )	$585,000$	50	$3,070,000$

$$\begin{aligned}\overline{\mathbf{x}} &= \frac{\mathbf{fx}}{\mathbf{f}} \\ &= \frac{3,070,000}{50} \\ &= 61,400\end{aligned}$$

Therefore, the mean is $61,400$

b) The modal class is $50,000 - 59,999$ **KES**, since is the one with the highest frequency.

□

Example 3.1.31 The data below represents the times (in seconds) recorded in the heats of a 100 **m** race during an athletics event:

$14.5, 13.7, 14.8, 15.3, 15.1, 14.2, 14.9, 12.6, 11.9, 13.1, 12.3, 14.7, 14.1, 15.0,$

14.3, 15.2, 11.7, 12.9, 13.5, 15.4, 12.8, 12.1, 14.4, 13.2, 14.6, 11.6, 12.7, 15.5
14.0, 14.9, 13.9, 12.0, 13.8, 15.2, 13.3

a) Create a frequency distribution table using class intervals:

- 11.5 – 11.9
- 12.0 – 12.4
- 12.5 – 12.9
- 13.0 – 13.4
- 13.5 – 13.9
- 14.0 – 14.4
- 14.5 – 14.9
- 15.0 – 15.4
- 15.5 – 15.9

b) Determine the modal class

c) Estimate median based on the frequency table.

d) Find the mean based on the frequency table.

Solution.

a) To create a frequency distribution table We need to count how many values fall into each class interval.

Table 3.1.32

Class Interval (seconds)	**Midpoint(x)**	**Frequency(f)**	**Cumulative frequency(CF)**	**fx**
11.5 – 11.9	11.7	2	2	23.4
12.0 – 12.4	12.2	4	6	36.6
12.5 – 12.9	12.7	5	11	38.
13.0 – 13.4	13.2	4	15	52.8
13.5 – 13.9	13.7	5	20	54.8
14.0 – 14.4	14.2	6	26	71.0
14.5 – 14.9	14.7	5	31	73.5
15.0 – 15.4	15.2	4	35	60.8
15.5 – 15.9	15.7	1	36	15.7
Total ( )	123.3	36	36	491

b) The modal class is the class interval with the highest frequency.

From the table, The highest frequency is 6, which appears in class interval: 14.0 – 14.4.

So, the modal class is 14.0 – 14.4

c) Total frequency (f) = 36

Median position is $\frac{36}{2} = 18$

The cumulative frequency just before 18 is 15, and the next class reaches 20, so the median class is 13.5 – 13.9.

Using the median formula:

$$\textbf{Median} = \mathbf{L} + \left(\frac{\frac{n}{2} - \mathbf{CF}}{\mathbf{F}}\right) \times \mathbf{C}$$

Where;

- L = 13.5 (lower boundary of median class)
- $\frac{n}{2} = 18$
- CF = 15 (cumulative frequency before the median class)
- F = 5 (frequency of the median class)
- C = 0.5 class width

$$\begin{aligned}\textbf{Median} =&13.5 + (\frac{18-15}{5}) \times 0.5\\ =&13.5 + (\frac{3}{5} \times 0.5)\\ =&13.5 + 0.3\\ =&13.8\end{aligned}$$

Thus, the median time is 13.8 **seconds**.

d) The mean is given by;

$$\overline{\mathbf{x}} = \frac{\mathbf{fx}}{\mathbf{f}}$$

From the table;
fx = 491.2
f = 36

$$\begin{aligned}=&\frac{491.2}{36}\\ =&13.64\end{aligned}$$

Thus, the mean time is 13.64 **seconds**

□

Exercises

1. Mathematics test scores for 60 students in a class from Ichina primary school are:
$32, 45, 12, 56, 38, 74, 60, 29, 41, 50, 55, 67, 72, 31, 47,$
$39, 18, 26, 64, 42, 48, 52, 69, 77, 35, 58, 23, 19, 61, 54,$
$70, 33, 28, 37, 44, 46, 30, 49, 79, 62, 21, 16, 53, 57, 40,$
$34, 25, 68, 66, 51, 59, 71, 27, 20, 36, 43, 63, 65, 75, 80$

a) Create a frequency distribution table using class intervals of 10, starting from 10 − 19, 20 − 29, 30 − 39, ..., 70 − 79.

b) Determine the modal class

c) Estimate the mean and median from the frequency table.

2. The population of 50 towns in Kakamega was recorded as follows:
$152, 168, 140, 155, 172, 184, 176, 193, 150, 160, 175, 143, 182, 164, 149,$

$170, 185, 157, 169, 188, 154, 178, 166, 147, 190,$
$145, 180, 158, 137, 174, 192, 141, 165, 187, 144,$
$162, 153, 171, 139, 148, 156, 183, 177, 186, 159$

a) Create a grouped frequency table with class intervals of 10, starting from $135 - 144$.

b) Determine the modal class.

c) Estimate the mean and median from the distribution.

3. The monthly electricity bills (in KES) of households in a town are recorded in the table below:

Table 3.1.33

Electricity Bill (KES)	**Frequency (f)**
$1,000 - 1,999$	6
$2,000 - 2,999$	10
$3,000 - 3,999$	14
$4,000 - 4,999$	12
$5,000 - 5,999$	8

a) Find the median electricity bill.

b) Identify the modal class.

4. A researcher collects data on daily rainfall (in mm) over a month and organizes it into 10 equal class intervals

Table 3.1.34

Rainfall (mm)	**Frequency (f)**
$0 - 9$	2
$10 - 19$	4
$20 - 29$	6
$30 - 39$	8
$40 - 49$	10
$50 - 59$	12
$60 - 69$	9
$70 - 79$	6
$80 - 89$	4
$90 - 100$	2

a) Identify the modal class.

b) Calculate the Mean rainfall.

c) Determine the Median rainfall.

Checkpoint 3.1.35 A researcher collects data on daily rainfall (in mm) over a month and organises it into 10 equal class intervals as shown below.

Rainfall (mm)	Frequency
0 – 9	1
10 – 19	2
20 – 29	7
30 – 39	7
40 – 49	7
50 – 59	13
60 – 69	11
70 – 79	9
80 – 89	6
90 – 100	2

Analyze the rainfall distribution and determine the central tendency measures listed below.

a) Modal class: ______ b) Mean rainfall (mm): ______ c) Median rainfall (mm): ______

Enter the modal class as an interval in the form `[a,b]`. *Round numerical answers to 2 decimal places.*

Answer 1. $[50, 59]$

Answer 2. 54.5153846154

Answer 3. 56.5384615385

Solution. Worked Solution To calculate the statistics, we first need to complete the frequency distribution table with the Midpoint (x), Cumulative Frequency (CF), and the product of frequency and midpoint (fx).

Class Interval	Midpoint (x)	Frequency (f)	Cumulative Frequency (CF)	fx
0 – 9	4.5	1	1	4.5
10 – 19	14.5	2	3	29.0
20 – 29	24.5	7	10	171.5
30 – 39	34.5	7	17	241.5
40 – 49	44.5	7	24	311.5
50 – 59	54.5	13	37	708.5
60 – 69	64.5	11	48	709.5
70 – 79	74.5	9	57	670.5
80 – 89	84.5	6	63	507.0
90 – 100	95.0	2	65	190.0
Total (Σ):		65		3543.5

a) Modal Class

The modal class is the class interval with the highest frequency.

From the table, the highest frequency is 13.

Therefore, the modal class is $50 - 59$.***b) Median Rainfall***

First, determine the median position:Position $= \frac{N}{2} = \frac{65}{2} = \frac{65}{2}$ Looking at the *Cumulative Frequency (CF)* column, the value just before $\frac{65}{2}$ is 24, and the next class reaches 37.

Thus, the Median Class is $50 - 59$.

Now, apply the median formula:Median $= L + \left(\frac{\frac{N}{2} - CF}{f}\right) \times c$ Where:

- $L = 50$ (Lower boundary of the median class)
- $$\frac{N}{2} = \frac{65}{2}$$

- $CF = 24$ (Cumulative frequency before the median class)
- $f = 13$ (Frequency of the median class)
- $c = 10$ (Class width)

Median $= 50 + \left(\frac{\frac{65}{2}-24}{13}\right) \times 10$ Median $= 50 + \left(\frac{\frac{17}{2}}{13}\right) \times 10$ Median $\approx$ **56.54** mm

c) Mean Rainfall

The mean is calculated using the formula:$\bar{x} = \frac{\Sigma fx}{\Sigma f}$ From the table totals:

- $\Sigma fx = 3543.5$
- $\Sigma f = 65$

$\bar{x} = \frac{3543.5}{65}$ $\bar{x} \approx$ **54.52** mm

Checkpoint 3.1.36 Mathematics test scores for 40 students in a class are listed below:

[64, 23, 35, 51, 11, 19, 55, 38, 64, 68, 46, 26, 67, 40, 32, 52, 11, 51, 79, 22, 61, 19, 52, 41, 30, 55, 23, 59, 70, 78, 50, 53,

a) Complete the frequency distribution table below by counting the number of scores that fall into each class interval.

Class Interval	Frequency
10 − 19	______
20 − 29	______
30 − 39	______
40 − 49	______
50 − 59	______
60 − 69	______
70 − 79	______

b) Identify the Modal Class: ______

c) Using the table you created, calculate the following estimates (round to 2 decimal places):

- Mean Score: ______
- Median Score: ______

Answer 1. 5

Answer 2. 6

Answer 3. 5

Answer 4. 3

Answer 5. 10

Answer 6. 6

Answer 7. 5

Answer 8. [50 - 59]

Answer 9. 45.75

Answer 10. 51.0

Solution. ***a) Frequency Table Construction***

To fill the table, we count how many scores appear in each interval. It helps to sort the data first:

[11, 11, 19, 19, 19, 22, 23, 23, 24, 26, 27, 30, 32, 35, 38, 38, 40, 41, 46, 50, 51, 51, 52, 52, 53, 55, 55, 55, 59, 61, 63, 64,

Class	Midpoint (x)	Frequency (f)	fx	Cum. Freq (CF)
10-19	$\frac{29}{2}$	5	$\frac{145}{2}$	5
20-29	$\frac{49}{2}$	6	147	11
30-39	$\frac{69}{2}$	5	$\frac{345}{2}$	16
40-49	$\frac{89}{2}$	3	$\frac{267}{2}$	19
50-59	$\frac{109}{2}$	10	545	29
60-69	$\frac{129}{2}$	6	387	35
70-79	$\frac{149}{2}$	5	$\frac{745}{2}$	40
Total		40	1830	

b) Modal Class

The highest frequency is 10, which corresponds to the interval ***50 - 59***.

c) Estimated Mean and Median

Mean: $\bar{x} = \frac{\sum fx}{\sum f} = \frac{1830}{40} \approx \mathbf{45.75}$ ***Median:***

Position $= \frac{N}{2} = \frac{40}{2} = 20$. Looking at the CF column, the median lies in the 50 - 59 class. $Median = L + \left(\frac{\frac{N}{2} - CF_{prev}}{f_{med}}\right) \times h$ $Median = 50 + \left(\frac{20-19}{10}\right) \times 10$ $Median \approx \mathbf{51}$

3.1.4 Representation of Data

Representation of data is the process of presenting collected information (data) in an organized and visual form so that it is easy to understand, interpret and analyze.

Instead of leaving data as raw numbers, we use tables, charts and graphs to show patterns, trends, and comparisons more clearly.

3.1.4.1 Drawing Histograms and Frequency Polygons of Data

Activity 3.1.6 Work in groups

In a school with 500 students, their heights were measured and recorded in the following table.

Table 3.1.37

Height (cm)	**Number of Students (Frequency)**
$140-149$	30
$150-159$	70
$160-169$	110
$170-179$	150
$180-189$	90
$190-199$	50

1. Choose a suitable scale and represent the data on a histogram and a frequency polygon.
2. Compare and discuss your graphs with other groups.

Key Takeaway 3.1.38 A histogram uses adjacent bars to show frequency distribution, while a frequency polygon connects the midpoints of the bars with a line to show patterns.

Class Width is the difference between the upper and lower boundaries of a class.

Equal class width means all bars have the same width.

Unequal class width means bars have different widths to better represent uneven data.

Frequency density is a measure used in histograms to ensure that the area of each bar represents the actual frequency of observations, especially when class widths are unequal.

Frequency density is calculated using the formula:

$$\textbf{Frequency density} = \frac{\textbf{Frequency}}{\textbf{Class width}}$$

Where;
Frequency is the number of observations in a class interval.
Why Use Frequency Density Instead of Frequency?

- In a histogram, the area of each bar (not just the height) represents the frequency.
- If class widths are unequal, simply plotting frequency would distort the representation.
- Using frequency density ensures that the area of each bar remains proportional to the actual frequency.

Midpoint of a class interval represents the central value of that range. It is the average of the lower and upper boundaries of the class.

Formula for midpoint:

$$\textbf{Midpoint} = \frac{\textbf{Lower bound} + \textbf{Upper bound}}{2}$$

Example 3.1.39 The table below presents the salary distribution of employees in a company.

Table 3.1.40

Salary Range (KSh)	**Frequency**
1000 – 1500	42
1500 – 2000	35
2000 – 2500	20
2500 – 3000	15
3000 – 4000	18
4000 – 5000	42

Draw a histogram and a frequency polygon to represent the data.

Solution. To draw a histogram and a frequency polygon, we need to find the frequency density and the midpoint of each class interval.

We use frequency density instead of frequency to draw the histogram because the class widths are unequal.

The formula for frequency density:

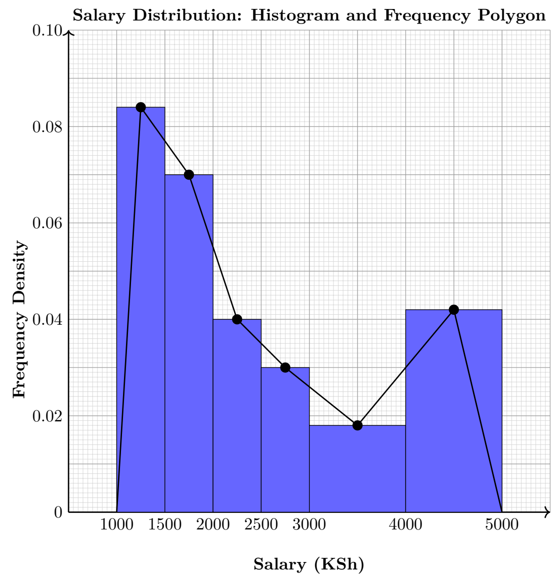
$$\textbf{Frequency density} = \frac{\textbf{Frequency}}{\textbf{Class width}}$$

The formula for Midpoint:

$$\textbf{Midpoint} = \frac{\textbf{Lower bound} + \textbf{Upper bound}}{2}$$

Table 3.1.41

Salary range(Ksh.)	Frequency	Class width	Frequency density	Midpoint
1000 – 1500	42	500	0.084	1250
1500 – 2000	35	500	0.070	1750
2000 – 2500	20	500	0.040	2250
2500 – 3000	15	500	0.030	2750
3000 – 4000	18	1000	0.018	3500
4000 – 5000	42	1000	0.042	4500



□

Example 3.1.42 The following frequency distribution shows the daily rainfall amounts (in mm) recorded at a weather station over a 60 day period.

Table 3.1.43

Rainfall(mm)	Frequency
0 – 5	22
6 – 10	15
11 – 15	12
16 – 25	8
26 – 40	3

Create a histogram to represent this data.

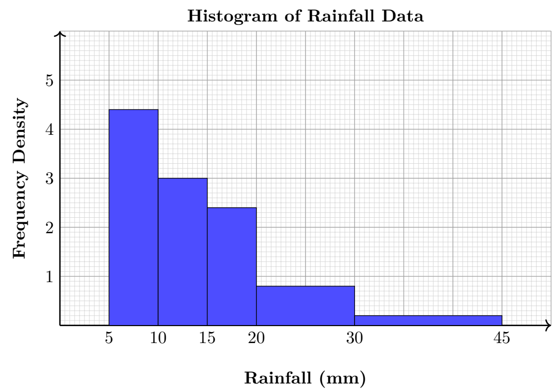
Solution. To draw a histogram we need to find the frequency density because class width are unequal.

The formula for frequency density:

$$\textbf{Frequency density} = \frac{\textbf{Frequency}}{\textbf{Class width}}$$

Table 3.1.44

Rainfall(mm)	Frequency	Class width	Frequency density
5 – 10	22	5	4.4
10 – 15	15	5	3.0
15 – 20	12	5	2.4
20 – 30	8	10	0.8
30 – 45	3	15	0.2



□

Exercises

1. The following data represents the heights (in cm) of 30 students in a class:
$150, 155, 160, 162, 165, 158, 170, 172, 168, 153, 163, 167, 175, 178, 161,$
$156, 169, 171, 159, 164, 173, 176, 157, 166, 174, 177, 154, 165, 179, 160$

a) Create a frequency table with class intervals of 5 **cm** and midpoints of the data.

b) Using the frequency table you created Draw a histogram to represent the data.

c) Draw a frequency polygon on the same axes as your histogram.

d) Label your axes and give your graphs appropriate titles.

2. The following data represents the ages of 25 people in a community meeting:
$20, 25, 30, 35, 40, 22, 28, 33, 38, 42, 27, 32, 37, 41,$
$24, 29, 34, 39, 43, 26, 31, 36, 44, 23, 45$

(a) Create a frequency table with class intervals of 5 years

(b) Draw a histogram to represent the data.

(c) Draw a frequency polygon on the same axes.

(d) Label the axes and provide titles for your graphs.

3. A school collected data on the number of books read by students in a term. The following frequency table shows the results:

Table 3.1.45

Number of Books Read	Frequency
0 – 2	15
3 – 5	25
6 – 8	35
9 – 11	15
12 – 14	10

(a) Draw a histogram to represent the data from your frequency table.

(b) On the same axes, draw a frequency polygon.

(c) Estimate the median number of books read. Explain your reasoning.

4. A survey was conducted to find out how much time people spend on social media daily. The following data was collected:

Table 3.1.46

Time (Minutes)	Frequency	Class Width	Frequency Density
0 – 10	15	10	1.5
10 – 20	25	10	2.5
20 – 30	30	10	3.0
30 – 60	40	30	1.33
60 – 120	20	60	0.33

a) Draw a histogram to represent the sales data.

b) Draw a frequency polygon to represent the sales data.

c) Label your axes and provide appropriate titles for your graphs.

Checkpoint 3.1.47 This question contains interactive elements.

Checkpoint 3.1.48 This question contains interactive elements.

3.1.5 Interpretation of data

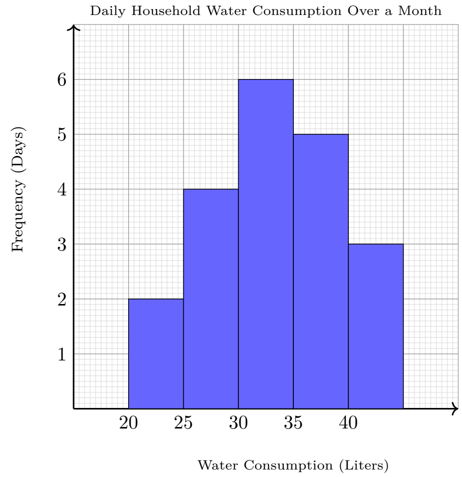
Interpretation of data is the process of examining and explaining the meaning of organized or represented data in order to draw conclusions, make decisions or solve problems.

Once data is collected and represented in tables, graphs or charts, we look for patterns, relationships and trends to understand what the data is telling us.

3.1.5.1 Interpreting Histograms and Frequency Polygons of Data

Activity 3.1.7 Work in groups

The histogram below represent a household's daily water consumption (in liters) recorded over a month.

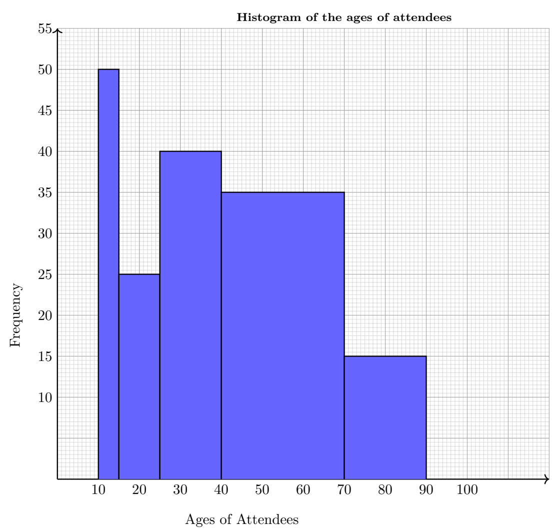


1. Determine the day when the water consumption was high.
2. Determine the day when the water consumption was low.
3. Discuss and share with other group.

Key Takeaway 3.1.49

- Interpretation of data helps us understand collected information by finding patterns, trends, and connections so we can make better decisions.

Example 3.1.50 The histogram below represents the ages of attendees recorded by the organizers at a community event.



a) How many age groups are represented in the histogram?

b) What is the total number of attendees recorded in the histogram?

c) Which age group has the highest number of attendees?

Solution.

a) By counting the number of bars in the histogram, we can determine the number of age groups.

The bars are 5

Therefore, there were five age groups that attended the event.

b) The total number of attendees is the sum of all frequencies (heights of the bars).

$50 + 25 + 40 + 35 + 15 = 165$

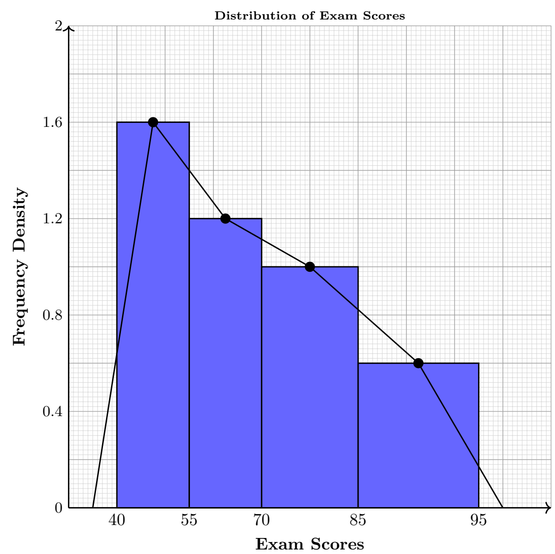
Therefore, the number of attendees were 165

c) The age group corresponding to the tallest bar has the highest number of attendees.

Therefore, the age group $10 - 15$ had the highest number of attendees, with a total of 50 participants.

□

Example 3.1.51 The graph below represents a histogram and frequency polygon of the distribution of exam scores of students in a Grade 10 class.



a) Describe the shape of the distribution of exam scores.

b) What is the midpoint of the class interval $70 - 85$?

c) Compare the height of the first bar ($40 - 55$ score range) to the height of the last bar ($85 - 95$ score range). What does this tell you about the number of students in those score ranges?

Solution.

a) The distribution is skewed to the right (positively skewed).

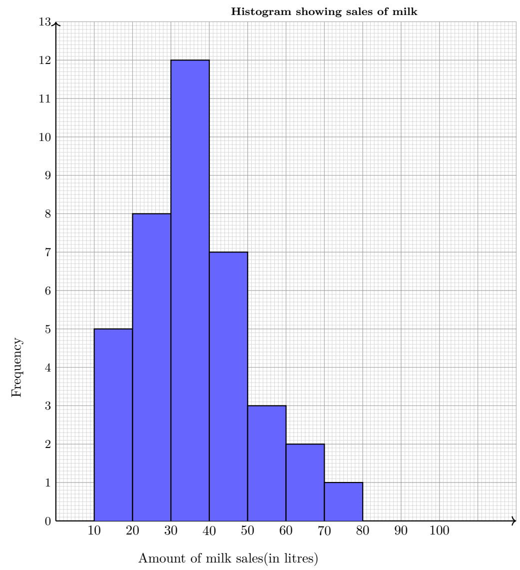
b) 77.5

c) The first bar is much taller than the last bar. This means that many more students got scores in the $40 - 55$ range than in the $85 - 95$ range.

□

Exercises

1. The following histogram shows sales of milk (in litres) sold by Akiru.



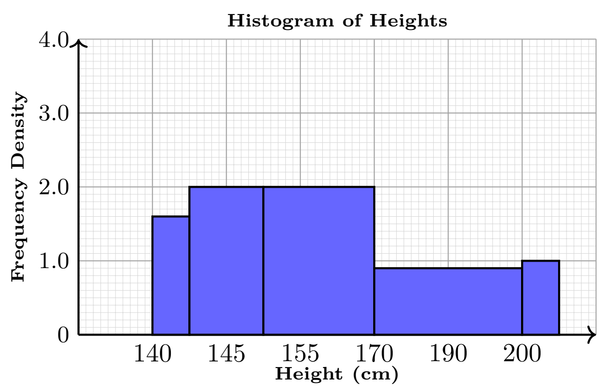
a) What does the y-axis represent?

b) What does the x-axis represent?

c) Which day did Akiru

d) Describe the shape of the histogram. Is it symmetrical or skewed? If skewed, is it skewed left or right?

2. The following histogram shows the height of students in a grade 10 class.



Use the information from the graph to answer the following questions:

a) Calculate the frequency of individuals with heights between 145 **cm** and 155 **cm** Show your working

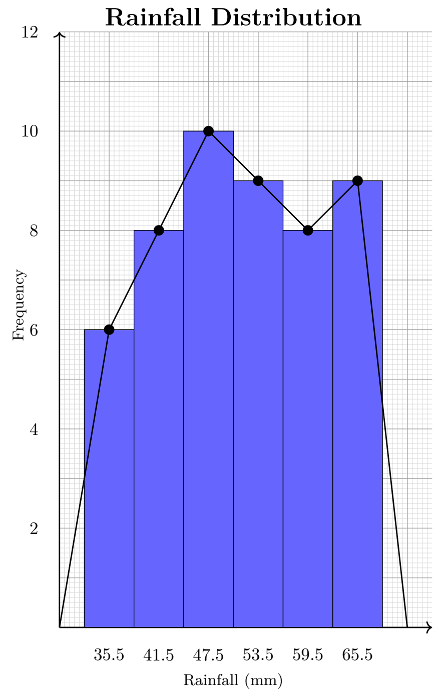
b) Identify the modal class.

c) Estimate the total number of individuals represented in the histogram.

d) Explain one difference between a histogram and a bar graph.

e) Describe the overall shape of the height distribution shown in the histogram.

3. Interpret the histogram and frequency polygon graph below and answer the questions given.



a) Describe the overall shape of this rainfall distribution graph.

b) At which rainfall ranges do the frequencies seem to decline?

c) Calculate the total frequency across all rainfall ranges.

d) What range of rainfall appears most frequently?

e) Estimate the median rainfall range from this distribution.

f) Which rainfall range appears to be the mode of this distribution?

g) What might cause variations in rainfall distribution?

Checkpoint 3.1.52 This question contains interactive elements.

Checkpoint 3.1.53 This question contains interactive elements.

3.2 Probability 1

3.2.1 Introduction to Probability

Activity 3.2.1 Work in groups

Write down 3 events that could happen today (e.g., "It will rain" or "I will be late to school")

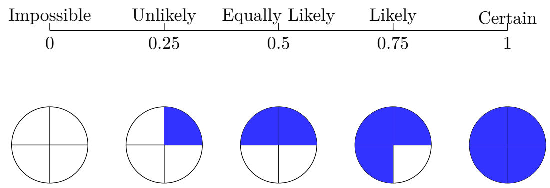
Predict the probability of each event: **Is it likely, unlikely, or certain**?

Key Takeaway

Probability is the measure of how likely an event is to occur. It is expressed as a number between 0 and 1, where:

- 0 means the event is impossible.
- 1 means the event is certain.
- A probability closer to 1 indicates a higher likelihood of the event occurring.

Probability is always between 0 and 1
Probability Scale



Key Terms in Probability

- Experiment - A process that leads to a specific result.
- Outcome - A possible result of an experiment.
- Event - A collection of one or more outcomes.
- Sample Space (S) - The set of all possible outcomes.
- Probability (P) - A measure of how likely an event is to occur.

Probability is widely used in everyday life, including:

- Weather Forecasting - Meteorologists predict the likelihood of rain based on past data.
- Sports - Coaches analyze the probability of winning based on past performance.
- Medicine - Doctors assess the probability of a patient responding to treatment.
- Finance and Insurance - Insurance companies use probability to determine policy pricing.
- Games of Chance - Dice rolling and card games use probability.

Checkpoint 3.2.1 Consider picking a blue pen from a bag that has no blue pens. Which of the following best represents the probability of this event?
[0 | Between 0 and 1 | 1]

Answer. 0

Solution. Worked Solution A probability of ***0*** represents an impossible event.
A probability of ***1*** represents a certain event.
A probability between ***0 and 1*** represents an uncertain event.
The correct probability value for this situation is ***0***.

Checkpoint 3.2.2 Consider the event of getting a number 7 when rolling a fair six-sided die. How would you classify this event?
[Certain | Possible | Impossible]

Answer. Impossible

Solution. Worked Solution An event that must happen is called ***certain***.
An event that cannot happen is called ***impossible***.
An event that may or may not happen is called ***possible***.

The correct classification of this event is ***Impossible***.

3.2.2 Probability Experiment and Sample Space

Activity 3.2.2 Work in groups

Here the material needed is only 1 coin

Toss a coin 20 times and record how many times it lands on heads and tails.

Discuss

- Give the sample space
- Compare your sample spaces

Key Takeaway

A probability experiment is an action or a process that results in a set of possible outcomes. The collection of all possible outcomes is called the sample space.

Example 3.2.3 A student of Makhokho Secondary School rolled a six-sided die. Give the sample space

Solution. the possible outcomes are

$$S = \mathbf{1,2,3,4,5,6}$$

the sample space consists of 6 possible outcomes. □

Example 3.2.4 Mutuse's bag contains 3 red ballgums, 2 blue ballgums, and 1 green ballgum.

If one ballgum is drawn at random, give the possible outcomes which is the probablity space

Solution.

$$S = \textbf{Red, Blue, Green}$$

The sample space consists of three possible outcomes. □

Exercises

1. If a day of the week is randomly selected, what is the sample space for the chosen day?

2. A restaurant offers three choices of main course: chicken, fish, or vegetarian. What is the sample space for the main course a customer might order?

3. A traffic light can show green, yellow, or red. What is the sample space for the possible colors the traffic light can display?

4. A store sells t-shirts in sizes small, medium, and large. What is the sample space for the t-shirt sizes sold?

5. A person spins a spinner with sections labeled A, B, C, and D. What is the sample space for the outcome of the spin?

Checkpoint 3.2.5 Consider Rolling a fair six-sided die once. Write the possible sample space for this experiment.

Answer. 1, 2, 3, 4, 5, 6

Solution. Worked Solution The sample space is the set of all possible outcomes of the experiment.

For this experiment, the sample space is:

$$1, 2, 3, 4, 5, 6$$

Checkpoint 3.2.6 Which one among the following is not an approach to the concept of Probability?

(1) Classical

(2) Axiomatic

(3) Exhaustive

(4) Relative frequency

Answer. (3)

Solution. Worked Solution There are three ways to assign probabilities to events: classical approach, relative-frequency approach, axiomatic approach.

Therefore ***Exhaustive*** is not an approach to the concept of Probability.

3.2.3 Probability of Simple Events

Activity 3.2.3 Work in groups

Kanyama rolls a fair six-sided die. What is the probability of Kanyama rolling a 4

1. Identify the Sample Space.
2. Identify the Favorable Outcomes
3. Apply the Probability Formula
4. Discuss and compare answers

Key Takeaway

A simple event is an event that consists of only one outcome in the sample space.

The probability of a simple event is given using the formula

$$\mathbf{P(E)} = \frac{\textbf{Number of favorable outcomes}}{\textbf{Number of Outcomes}}$$

where;

- $\mathbf{P(E)}$ is the probability of event $\mathbf{E}$
- Favorable outcomes refer to the specific event we are interested in
- Total outcomes refer to all possible outcomes in the sample space

Example 3.2.7 A bag contains 5 red balls and 3 blue balls. If one ball is picked at random, what is the probability that it is red?

Solution. Total number of balls $\mathbf{= 5 + 3 = 8}$

Number of red balls $\mathbf{= 5}$

Given a bag with 5 red balls and 3 blue balls, the possible outcomes when picking one ball are

$\mathbf{S = Red,Blue}$

Total outcomes $\mathbf{= 5 + 3 = 8}$

Probability of drawing a red ball is given by:

$\mathbf{P(Red)} = \frac{\textbf{Number of favorable outcomes}}{\textbf{Number of Outcomes}} = \frac{5}{8} = \mathbf{0.625}$

the probability of picking a red ball is 0.625 or 62.5% □

Example 3.2.8 A teacher at Sironga Secondary school randomly selects a student from a class of 30 students. If there are 12 girls and 18 boys in the class, what is the probability that the selected student is a girl?

Solution.

i). Sample Space is

$$\mathbf{S = Girl, Boy}$$

ii). The number of favorable outcomes that is choosing a girl = **12**

iii). Now, Applying our formula gives

$$\mathbf{P(Girl)} = \frac{\textbf{Number of girls}}{\textbf{Total number of students}}$$

$$= \frac{12}{30}$$

$$= \mathbf{0.4}$$

The probability of selecting a girl is 0.4 or 40% □

Exercises

1. What is the probability of selecting the letter 'a' from the name "Mukabwa"?

2. A deck of standard playing cards has 52 cards. What is the probability of drawing the 5 of Hearts?

3. A bag has 3 yellow marbles, 5 black marbles, and 2 white marbles. What is the probability of selecting a white marble?

4. A month is selected at random from a year. What is the probability that it is June?

5. A coin is tossed. What is the probability of getting tails?

6. A box contains tickets numbered from 1 to 10. What is the probability of drawing a ticket with the number 7?

7. A class has 25 students, and one student is chosen at random. What is the probability that a specific student is chosen?

8. What is the probability of selecting the letter "e" from the word "elephant"?

Checkpoint 3.2.9
What is the probability of picking a number less than 7 from a deck of cards?
Note: Enter your value corrected to two decimal places

____

Answer. 0.38

Checkpoint 3.2.10 James, a Grade 10 learner states that the probability of it raining today is 0.6. Is this value a valid probability?
[Yes | No]

Answer. Yes

Solution. Worked Solution The value of probability must lie between 0 and 1 inclusive.

Since the given value is 0.6, the correct response is ***Yes***.

3.2.4 Mutually Exclusive and Independent Events

3.2.4.1 Mutually Exclusive

Activity 3.2.4 Work in groups

1. Define Mutually exclusive events
2. State one example of Two events that are mutually exclusive
3. In a class of 40 students, 18 take French, 22 take German, and no student takes both.
   a). What is the probability that a randomly selected student takes French or German?
   b). Are the events “taking French” and “taking German” mutually exclusive? Explain.
4. Compare and discuss answers with other groups

Key Takeaway
Two events are mutually exclusive if they cannot occur at the same time. This means that if one event happens, the other cannot.
If **A** and B are mutually exclusive events, then;

$$\mathbf{P(A\ and\ B) = 0}$$

$$\mathbf{P}(\mathbf{A} \cap \mathbf{B}) = 0$$

The probability of either **A** or **B** occurring is;

$$\mathbf{P(A\ or\ B) = P(A) + P(B)}$$

Example

- A traffic light being red and the same traffic light being green at the exact same time.
- You are sleeping and you are wide awake at the exact same moment.
- A door being open and the same door being closed at the same time.

Example 3.2.11 Roll a fair six-sided die, what is the probability of rolling either a 3 or a 5?

Solution. Sample Space

$$\mathbf{S = 1,\ 2,\ 3,\ 4,\ 5,\ 6}$$

Favorable Outcomes

- $\mathbf{P(3)} = \frac{1}{6}$
- $\mathbf{P(5)} = \frac{1}{6}$

Since rolling a 3 and rolling a 5 are mutually exclusive events;

$$\mathbf{P(3\ or\ 5) = P(3) + P(5)}$$

$$= \frac{1}{6} + \frac{1}{6}$$

$$= \frac{2}{6} = \frac{1}{3}$$

the probability of rolling a 3 or 5 is $\frac{1}{3}$ or 33.33% □

Example 3.2.12 A card is drawn from a standard deck of 52 playing cards. Let Event A be drawing a Heart and Event B be drawing a Spade. Are these events mutually exclusive? Explain

a). Find **P(A) and P(B)**

b). Calculate **P(A or B)**

c). What is $\mathbf{P}(\mathbf{A} \cap \mathbf{B})$

Solution. Two events are mutually exclusive if they cannot happen at the same time. Since a card cannot be both a Heart and a Spade, the events are mutually exclusive.

1. We calculate the probability of drawing a Heart or a Spade.

   Since there are **13 Hearts** in the deck,

$$\mathbf{P(A)} = \frac{13}{52}$$

   Since there are **13 Spades** in the deck,

$$\mathbf{P(B)} = \frac{13}{52}$$

2. For mutually exclusive events, we use,

$$\mathbf{P(A\ or\ B) = P(A) + P(B)}$$

$$= \frac{13}{52} + \frac{13}{52}$$

$$= \frac{26}{52}$$

$$= \frac{1}{2}$$

3. Since these events are mutually exclusive, their intersection is zero:

$$\mathbf{P}(\mathbf{A} \cap \mathbf{B}) = 0$$

□

Exercises

1. A student at Khungu Senior Secondary School tosses a coin once. Are the events ”getting heads” and ”getting tails” mutually exclusive?

2. A person selects one piece of fruit from a bowl containing apples, bananas, and oranges.Is selecting an apple and selecting a banana mutually exclusive events?

3. A student recorded their method of transport to school for 10 days. The methods used were:

Walk, Bus, Bus, Walk, Bike, Walk, Bus, Bike, Walk, Bus

Let Event A = ”The student walked to school” and Event B = ”The student took the bus to school”

(a). Are events A and B mutually exclusive? Explain your answer.

(b). What is the probability that the student either walked or took the bus to school on a randomly chosen day?

4. A card is drawn from a standard 52-card deck.

(a). Are the events ”drawing a diamond” and ”drawing a club” mutually exclusive?

(b). What is the probability of drawing either a diamond or a club?

5. In a lottery competition, there are five cards labelled A, B, C, D, and F. A player must pick only one card to enter the competition.

(a). Are the events ”picking card A” and ”picking card F” mutually exclusive?

(b). What is the probability of either picking card A or picking card F?

3.2.4.2 Independent Events

Activity 3.2.5 Work in groups

1. Define Independent events
2. State one example of two events that are independent
3. A student can choose to join either the Science Club or the Drama Club, but not both.
   a). If the probability of joining Science Club is 40% and Drama Club is 30% , what is the probability that a student joins either club?
   b). Are these events mutually exclusive or independent? Explain.
4. Compare and discuss answers with other groups

Key Takeaway

Two events are independent if the occurrence of one does not affect the probability of the other occurring.

If A and B are independent events, then;

$$P(\mathbf{A\ and\ B}) = \mathbf{P(A)} \times \mathbf{P(B)}$$

For events **A** and **B**

$$\mathbf{P(A \cap B) \ = \ P(A) \times P(B)}$$

This means the probability of both events occurring together is the product of their individual probabilities.

Example

- A student bringing a lunch from home and another student buying a lunch from the cafeteria.
- A student answering a question correctly in english class and another student dropping their pencil in science class.

Example 3.2.13 A coin is flipped, and a six-sided die is rolled. What is the probability of getting heads and rolling a 6?

Solution.

i. the Sample Space is

   Possible coin outcomes **H, T**

   Possible die outcomes **1, 2, 3, 4, 5, 6**

ii. Favorable Outcomes

   $\mathbf{P(H)} = \frac{1}{2}$

   $\mathbf{P(6)} = \frac{1}{6}$

iii. Since flipping the coin and rolling the die are independent events

$$P(\mathbf{H\ and\ 6}) = \mathbf{P(H)} \times \mathbf{P(6)}$$

$$= \frac{1}{2} \times \frac{1}{6}$$

$$= \frac{1}{12}$$

the probability of getting heads and a 6 is $\frac{1}{12}$ or 8.33% □

Example 3.2.14 The probability that it rains on a given day is 40%, and the probability that a person is late to work is 20%.

Let $\mathbf{P(R)}$ represent the probability that it rains and $\mathbf{P(L)}$ be the probability that the person is late. The compliment of an event is the probability that it does not happen. Therefore, $\mathbf{P}(\mathbf{R}^c)$ will represent the probability that it does not rain and $\mathbf{P}(\mathbf{L}^c)$ the probability that the person is not late.

a). Find the probability that it rains $\mathbf{P(R)}$.

b). Find the probability that the person is late $\mathbf{P(L)}$.

c). Find the probability that it rains and the person is late.

d). Find the probability that it does not rain but the person is late.

e). Find the probability that it rains or the person is late.

Solution.

1. The probability that it rains is

$$\mathbf{P(R) = 0.4}$$

The probability that it rains is **0.4 (or 40%)**.

2. The probability that the person is late is

$$\mathbf{P(L) = 0.2}$$

The probability that the person is late is 0.2 or (20%).

3. Since rain and being late are independent,

$$\mathbf{P(R \cap L) = P(R) \times P(L)}$$

$$\mathbf{P(R \cap L)} = 0.4 \times 0.2 = 0.08$$

The probability that it rains and the person is late is **0.08 (or 8%)**.

4. The probability that it does not rain is

$$\mathbf{P(R}^c) = \mathbf{1 - P(R)} = 1 - 0.4 = 0.6$$

Since rain and being late are independent

$$\mathbf{P(R}^c \cap L) = \mathbf{P(R}^c) \times \mathbf{P(L)}$$

$$\mathbf{P(R}^c \cap L) = 0.6 \times 0.2 = 012$$

The probability that it does not rain but the person is late is 0.12 or (12%).

5. The probability that it rains or the person is late is

$$\mathbf{P(R \cup L) = P(R) + P(R) - P(R \cup L)}$$

$$\mathbf{P(R \cup L)} = 0.4 + 0.2 - 0.08$$

$$= 0.52$$

The probability that it rains or the person is late is 0.52 or (52%).

□

Exercises

1. A coin is tossed twice. What is the probability of getting heads on both tosses?

2. A die is rolled, and a coin is tossed. What is the probability of rolling a 6 on the die and getting tails on the coin?

3. A bakery in Makongeni produces cakes. The probability that a cake is decorated with chocolate icing is 0.7. If two cakes are made independently, what is the probability that both cakes are decorated with chocolate icing?

4. A seed has a 60% chance of germinating. If two seeds are planted independently, what is the probability that both seeds germinate?

Checkpoint 3.2.15 Assume that a noisy channel independently transmits symbols, say 0s 38% of the time and 1s 62% of the time. At the receiver, there is a 3% chance of obtaining any particular symbol distorted. What is the probability of receiving a 1, irrespective of which symbol is transmitted?

_______________ (Give your answer in two decimal places)

Answer. 0.61

Solution. Worked solution

Given that;

Probability 0s transmitted, $P(0)$ is 0.38

Probability 1s transmitted, $P(1)$ is 0.62$P(1\,\text{received}\,|\,0\,\text{transmitted})$, $P(1|0)$ is 0.03 Hence, the probability of receiving a zero is

$$\begin{aligned}
P(1) &= P(1\,\text{received}) \\
&= P(1|0)\cdot P(0) + P(1|1)\cdot P(1) \\
&= (0.03)\cdot(0.38) + (1-0.03)\cdot(0.62) \\
&= (0.03\cdot 0.38) + (0.97\cdot 0.62) \\
&= 0.61
\end{aligned}$$

Checkpoint 3.2.16 An event B and its complement B' are mutually exclusive. Is this statement correct?

(1) No

(2) Yes

Answer. (2)

Solution. Worked solution Two events C and D are mutually exclusive if they do not occur at the same time, i.e. $C \cap D = \emptyset$

By definition, an event B and its compliment B' have the following properties:

1. Additivity: $B \cup B' = S$ where S is the entire sample space, and thus $P(B \cup B') = 1$

2. Exclusivity: $B \cap B' = \emptyset$ and thus $P(B \cap B') = 0$

This means that in particular, by Property 2, B and B' are mutually exclusive.

Checkpoint 3.2.17 On a firing range, a rifleman has two attempts to hit a target. The probability of hitting the target with the first shot is $\frac{1}{2}$ and the probability of hitting with the second shot is $\frac{1}{5}$.The probability of hitting the target with both shots is $\frac{1}{10}$.

Find the probability of the following events below.
M : "Missing the target with both shots".
$P(M) =$________
N : "Hitting with the first shot and missing with the second".
$P(N) =$________

Answer 1. $\frac{2}{5}$

Answer 2. $\frac{2}{5}$

Solution. ***Worked solution***

Find the probability of the following events below.Find P(M) with M : "Missing the target with both shots".

To answer to this question, we denote the events F, S and H as follows:
F : "Hitting the target with the first shot".
S : "Hitting the target with the second shot".
H : "Hitting the target with both shots".

We get from the information provided by the text question that $P(F) = \frac{1}{2}$, $P(S) = \frac{1}{5}$ and $P(H) = \frac{1}{10}$.

Then $P(M) = 1 - P(F \cup S)$, and $P(F \cup S) = (P(F) - P(H)) + (P(S) - P(H)) + P(H)$.

So $P(F \cup S) = (\frac{1}{2} - \frac{1}{10}) + (\frac{1}{5} - \frac{1}{10}) + \frac{1}{10}$.

Thus $P(F \cup S) = \frac{3}{5}$, and $P(M) = 1 - \frac{3}{5}$.

Therefore $P(M) = \frac{2}{5}$.Find P(N) with N :"hitting with the first shot and missing with the second

We have: $P(N) = P(F) - P(H)$. Then
$P(N) = \frac{1}{2} - \frac{1}{10}$
Therefore, $P(N) = \frac{2}{5}$

3.2.5 Law of Probability

3.2.5.1 Addition Law of Probability

Activity 3.2.6 Work in groups

The probability that a student passes Mathematics is 75% and the probability that they pass English is 60%. If the probability of passing both is 50%, find the probability that the student passes either Mathematics or English.

compare answers with other groups

Key Takeaway

The addition law is used to find the probability of either one event or another occurring.

Mutually exclusive events

The probability of either event occurring is;

$$\mathbf{P(A \cup B) = P(A) + P(B)}$$

$$\mathbf{P(A\ or\ B) = P(A) + P(B)}$$

Since mutually exclusive events cannot happen at the same time, $\mathbf{P(A \cap B)} = 0$

Non-mutually exclusive events

If two events can happen at the same time, we must subtract the probability of them happening together to avoid double counting.

That is;

$$\mathbf{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

P(A or B) = P(A) + P(B) - P(A and B)

Example 3.2.18 A standard deck has 52 cards, with 13 hearts and 13 clubs.

Since a single card cannot be both a heart and a club, the events are mutually exclusive.

Solution.

- $P(\textbf{Heart}) = \frac{13}{52}$
- $P(\textbf{Club}) = \frac{13}{52}$

$$P(\textbf{Heart or Club}) = \frac{13}{52} + \frac{13}{52}$$

$$= \frac{26}{52} = \frac{1}{2}$$

the probability of drawing either a heart or a club is 0.5 or 50% □

Example 3.2.19 A standard deck has

- 26 red cards
- 4 kings (2 of them are red)

now,

$$P(\textbf{Red}) = \frac{26}{52}$$

$$P(\textbf{Heart}) = \frac{13}{52}$$

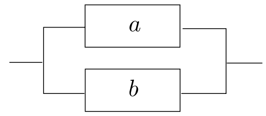
$$P(\textbf{Red and Heart}) = \frac{13}{52}$$

$$P(\textbf{Red or Heart}) = \frac{26}{52} + \frac{13}{52} - \frac{13}{52} = \frac{26}{52}$$

$$= \frac{1}{2}$$

the probability of drawing either a red card or a king is 0.50 or 50% □

Checkpoint 3.2.20 The diagram shows a simplified circuit in which two independent components a and b are connected in parallel.



The circuit functions if either or both of the components are operational. It is known that if A is the event 'component a is operating' and B is the event 'component b is operating' then $\mathrm{P}(A) = 0.95$, $\mathrm{P}(B) = 0.97$ and $\mathrm{P}(A \cap B) = 0.94$. Find the probability that the circuit is functioning.

____

Answer. 0.98

Solution. The probability that the circuit is functioning is $\mathrm{P}(A \cup B)$. In words: either a or b or

both must be functioning if the circuit is to function. Using the keypoint:

$$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B) \tag{3.2.1}$$

$$= 0.95 + 0.97 - 0.94 = 0.98 \tag{3.2.2}$$

Checkpoint 3.2.21 A bag contains 20 balls, 3 are coloured red, 5 are coloured green, 2 are coloured blue, 4 are coloured white and 6 are coloured yellow. One ball is selected at random. Find the probabilities of the following events.

1. The ball is either red or green. _____
2. The ball is not blue. _____
3. The ball is either red or white or blue. (Hint: consider the complementary event.) _____

Answer 1. $\frac{2}{5}$

Answer 2. $\frac{9}{10}$

Answer 3. $\frac{9}{20}$

Solution. A bag contains 20 balls, 3 are coloured red, 5 are coloured green, 2 are coloured blue, 4 are coloured white and 6 are coloured yellow.

Note that a ball has only one colour, designated by the letters R, G, B, W, Y.

1. $\mathrm{P}(R \cup G) = \mathrm{P}(R) + \mathrm{P}(G) = \dfrac{3}{20} + \dfrac{5}{20} = \dfrac{2}{5}$.
2. $\mathrm{P}(B') = 1 - \mathrm{P}(B) = 1 - \dfrac{2}{20} = \dfrac{9}{10}$.
3. The complementary event is $G \cup Y$, $\mathrm{P}(G \cup Y) = \dfrac{5}{20} + \dfrac{6}{20}$.

   Hence $\mathrm{P}(R \cup W \cup B) = 1 - \mathrm{P}(G \cup Y) = \dfrac{9}{20}$.

In part c) we could alternatively have used an obvious extension of the law of addition for mutually exclusive events:

$$\mathrm{P}(R \cup W \cup B) = \mathrm{P}(R) + \mathrm{P}(W) + \mathrm{P}(B) = \frac{3}{20} + \frac{2}{20} + \frac{4}{20} = \frac{9}{20}.$$

Checkpoint 3.2.22 A box contains 3 bad tube(s) and 7 good tubes. Two are drawn out together. One of them is tested and found to be good. What is the probability that the other one is also good?

_____

Answer. $\frac{2}{3}$

Solution. A box contains 3 bad tube(s) and 7 good tubes, so, in total 10 tubes.

Let $G_i = \{i^{th}$ tube is good$\}$.

The probability to draw one good tube out of the box is $G_1 = \dfrac{7}{10}$.

If the first tube was good, then when the second tube is selected there are only 6 good tubes left in the box which now contains 9 tubes.

Hence, $P(G_2|G_1) = \dfrac{2}{3}$.

Exercises

1. A student can get an A, B, C, D, or F in a class. What is the probability that the student gets an A or a B?

2. A die is rolled. What is the probability of rolling a 1 or a 6?

3. In a class of 30 students, 15 students like math, 10 students like chemistry, and 5 students like both math and chemistry. What is the probability that a randomly chosen student likes math or chemistry?

4. A bag contains 8 blue marbles and 5 yellow marbles. What is the probability of drawing a blue marble or a yellow marble?

5. In a class of 25 students, 12 play soccer, 10 play basketball, and 5 play both. What is the probability that a randomly chosen student plays soccer or basketball?

6. A bag contains letters of the word $MATHEMATICS$. What is the probability of selecting a vowel or the letter M?

7. A number is chosen between 1 and 10. What is the probability that it is a 3 or a 7?

8. A day of the week is chosen at random. What is the probability that it is a Saturday or a Sunday?

3.2.5.2 multiplication rule

Activity 3.2.7 Work in groups

A factory produces 90% good items and 10% defective items. A quality check is performed on two randomly selected items

a). Find the probability that both items are good.

b). Find the probability that at least one item is defective.

c). Are these events independent? Explain.

Key Takeaway

The multiplication rule is used to find the probability of two events happening together.

For Independent events,the probability of both occurring is

$$\mathbf{P(A\ and\ B) = P(A) \times P(B)}$$

For Dependent events

$$\mathbf{P(A \cap B) = P(A) \times P(B|A)}$$

Here, $\mathbf{P(B|A)}$ is the probability that B happens given that A has already occurred.

Example 3.2.23 A student in Modegashe primary school was instructed to roll a Die and Toss a Coin.

What was the probability of rolling a 4 on the die and getting head on the coin?

Solution.

i. Probability of rolling a 4 on a six-sided die is

$$\frac{1}{6}$$

ii. Probability of getting heads on the coin is

$$= \frac{1}{2}$$

$$\textbf{so, } \mathbf{P(4 \text{ and } H)} = \frac{1}{6} \times \frac{1}{2}$$

$$= \frac{1}{12}$$

the probability of rolling a 4 and flipping heads is $\frac{1}{12}$ or 8.33% □

Example 3.2.24 A person has a 60% probability of catching the first bus and an 80% probability of catching the second bus (if they miss the first one)

a). Find the probability that the person catches the first bus.

b). Find the probability that the person misses the first bus but catches the second.

c). Find the probability that the person misses both buses.

Solution. Probability of catching the first bus:

$$\mathbf{P(A) = 0.6}$$

Probability of missing the first bus

$$\mathbf{P}(\mathbf{A}^c) = 1 - \mathbf{P(A)} = 1 - 0.6 = 0.4$$

Probability of catching the second bus, given that the first bus was missed

$$\mathbf{P}(\mathbf{B}|\mathbf{A}^c) = 0.8$$

Probability of missing the second bus, given that the first bus was missed

$$\mathbf{P}(\mathbf{B}^c|\mathbf{A}^c) = 1 - \mathbf{P}(\mathbf{B}|\mathbf{A}^c) = 1 - 0.8 = 0.2$$

a). The probability of catching the first bus is directly given as

$$\mathbf{P(A)} = 0.6$$

The probability of catching the first bus is0.6 or 60%.

b). Missing the first bus $\mathbf{A}^c$

Catching the second bus B

Since these events are dependent, we use the multiplication rule

$$\mathbf{P}(\mathbf{A}^c \cap \mathbf{B}) = \mathbf{P}(\mathbf{A}^c) \times \mathbf{P}(\mathbf{B}|\mathbf{A}^c)$$

$$\mathbf{P}(\mathbf{A}^c \cap \mathbf{B}) = 0.4 \times 0.8 = 0.32$$

The probability of missing the first bus but catching the second is 0.32 or 32%.

c). Missing the first bus $\mathbf{A}^c$

Missing the second bus $\mathbf{B}^c$

Again, using the multiplication rule

$$\mathbf{P}(\mathbf{A}^c \cap \mathbf{B}^c) = \mathbf{P(A)}^c \times \mathbf{P}(\mathbf{B}^c|\mathbf{A}^c)$$

$$\mathbf{P}(\mathbf{A}^c \cap \mathbf{B}^c) = 0.4 \times 0.2 = 0.08$$

The probability of missing both buses is 0.08 or 8%.

□

Checkpoint 3.2.25 Multiplication Rule. Load the question by clicking in the button below.

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Exercises

1. A coin is tossed twice. What is the probability of getting heads on both tosses?

2. A die is rolled, and a coin is tossed. What is the probability of rolling a 6 and getting tails?

3. A weather forecast predicts a 60% chance of sunshine on Monday and a 70% chance of sunshine on Tuesday. Assuming these forecasts are independent, what is the probability of sunshine on both Monday and Tuesday?

4. A farmer in Gakuonyo plants two seeds. Each seed has a 75% chance of germinating. What is the probability that both seeds germinate?

3.2.6 Tree Diagrams and Independent Events

Activity 3.2.8 Work in groups

A student at Kenyaoni Senior School flips a coin and then spins a spinner with two equal sections, **Yes** and **No**.

i. Draw a tree diagram to represent the possible outcomes.

ii. What is the probability that the coin lands on heads and the spinner lands on No?

Key Takeaway

A tree diagram is a visual representation of all possible outcomes of an event.

It helps in organizing complex probability problems, especially for events that happen in sequence.

Example 3.2.26 A fair coin is tossed twice.

a). Draw a tree diagram showing all possible outcomes.

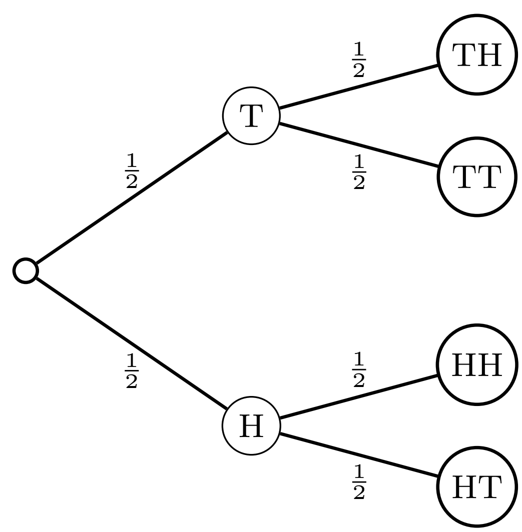
b). What is the probability of getting exactly one head?

c). What is the probability of getting at least one tail?

d). What is the probability of getting two heads?

Solution.

a.) Tree diagram showing all possible outcomes.



b.) Probability of getting exactly one head

The favorable outcomes are **HT and TH**.

$$\mathbf{P(HT) + P(TH)}$$

$$\frac{1}{4} + \frac{1}{4} = \frac{2}{4}$$

$$\frac{2}{4} = \frac{1}{2}$$

c.) Probability of getting at least one tail

The favorable outcomes are **HT, TH, TT (all outcomes except HH)**.

$$\mathbf{P(at\ least\ one\ tail) = P(HT) + P(TH) + P(TT)}$$

$$\frac{1}{4} + \frac{1}{4} + \frac{1}{4}$$

$$= \frac{3}{4}$$

d.) Probability of getting two heads

Only one outcome satisfies this condition: HH.

$$\mathbf{P(HH)} = \frac{1}{4}$$

□

Example 3.2.27 A bag contains 3 red and 2 blue balls. A ball is drawn without replacement.

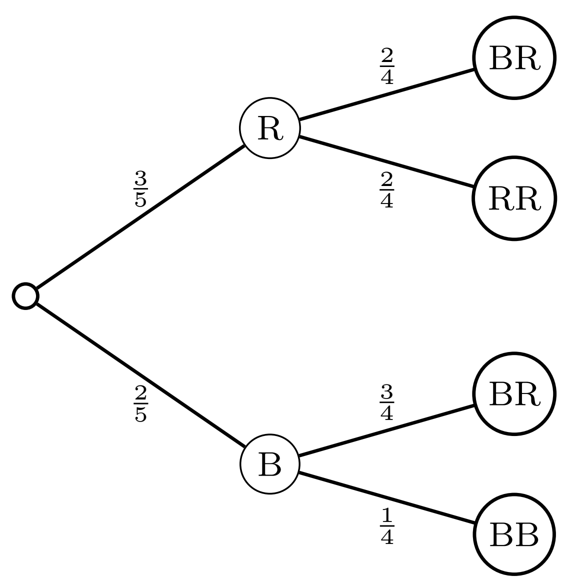
a). Draw a tree diagram showing the possible outcomes.

b). What is the probability of drawing a red ball followed by a blue ball?

c). What is the probability of drawing two red balls?

d). What is the probability of drawing at least one blue ball?

Solution. Here is the tree diagram



Assign Probabilities

a). The probability of drawing a red ball first is

$$\mathbf{P(R)} = \frac{3}{5}$$

- If the first ball is red, the probability of drawing a red ball second is;

$$\mathbf{P(R/R)} = \frac{2}{4} = \frac{1}{2}$$

- If the first ball is red, the probability of drawing a blue ball second is

$$\mathbf{P(B/R)} = \frac{2}{4} = \frac{1}{2}$$

The probability of drawing a blue ball first is:

$$\mathbf{P(B)} = \frac{2}{5}$$

- If the first ball is blue, the probability of drawing a red ball second is:

$$\mathbf{P(R/B)} = \frac{3}{4}$$

- If the first ball is blue, the probability of drawing another blue ball is:

$$\mathbf{P(B/B)} = \frac{1}{4}$$

b). Probability of drawing a red ball followed by a blue ball

Favorable outcome: RB

$$\mathbf{P(RB)} = \mathbf{P(R)} \times \mathbf{P(B|R)}$$

$$= \frac{3}{5} \times \frac{2}{4} = \frac{6}{20} = \frac{3}{10}$$

c). Probability of drawing two red balls

Favorable outcome: RR

$$\mathbf{P(RR)} = \mathbf{P(R)} \times \mathbf{P(R|R)}$$

$$= \frac{3}{5} \times \frac{2}{4} = \frac{6}{20} = \frac{3}{10}$$

d). Probability of drawing at least one blue ball

$$\text{P(at least one blue)} = \textbf{1 - P(two red)}$$

$$1 - \frac{3}{10} = \frac{7}{10}$$

□

Example 3.2.28 A student at a shop is choosing a meal and a drink. This is what is available

- **Meals** ; Bread (B), Andazi (A), Chapati (Ch)
- **Drinks** ; Juice (J), Soda (S)

The student buys one meal at one drink
what is the probability of ;

1. Choosing Andazi and Soda?
2. Choosing Juice as a drink?

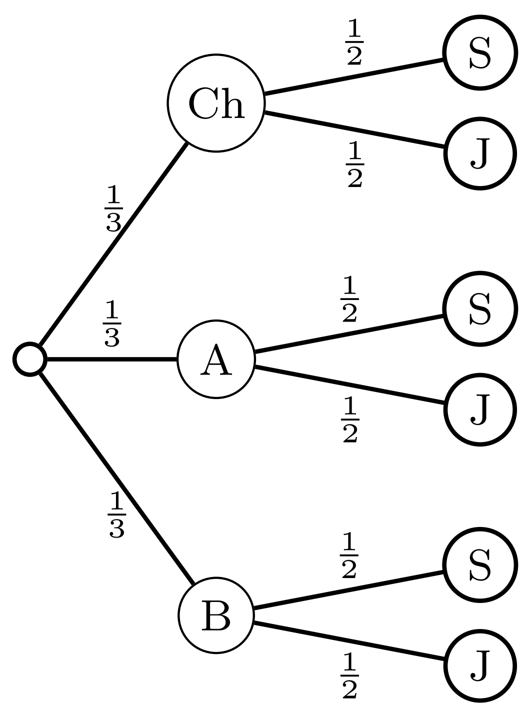
Solution.

1. The Sample Space where the possible meal-drink combinations are :

$$\textbf{S} = \textbf{(B,J), (B,S), (A,J), (A,S), (Ch,J), (Ch,S)}$$

There are **3 meals** × **2 drinks = 6 total choices**.

2. the Tree Diagram



3. Now we find the probabilities

   i. Probability of Choosing an Andazi and Soda

$$\mathbf{P(A\ and\ S)} = \frac{1}{3} \times \frac{1}{2}$$

$$= \frac{1}{6}$$

   ii. Probability of Choosing Juice as a Drink
   There are 3 favorable outcomes that is (B, J) , (A, J), (Ch, J) there are 3 out of 6 possible choices, so the probability is:

$$\mathbf{P(J)} = \frac{3}{6} = \frac{1}{2}\ \textit{or}\ \mathbf{50}\%$$

□

Exercises

using Tree diagrams solve;

1. A coin is tossed three times. What is the probability of getting exactly two heads?

2. You have two bags. Bag 1 has 3 blue marbles and 1 red marble. Bag 2 has 2 blue marbles and 2 red marbles. You pick one marble from each bag. What is the probability of picking two blue marbles?

3. A store sells two types of phone cases. 30% are black, 70% are clear. Two phone cases are sold independently. What is the probability that one is black and the other is clear?

4. A student takes two true/false quizzes. What is the probability that they get both quizzes completely correct?

5. A teacher assigns homework on Monday and Tuesday. There's a 80% chance of homework on Monday and a 80% chance on Tuesday. What is the probability there is homework on both days?

6. A student has a 60% chance of completing their math homework on time and a 75% chance of completing their English homework on time. Assuming these events are independent, what is the probability that the student completes both assignments on time?

Checkpoint 3.2.29

1. A card is drawn at random from a deck of 52 playing cards. What is the probability that it is an ace or a picture card (i.e. K, Q, J)?_____

2. In a single throw of two 6-faced dice, what is the probability that neither a double nor a sum of 9 will appear? _____

Answer 1. $\frac{4}{13}$

Answer 2. $\frac{13}{18}$

Solution.

1.

$$F = \{\text{face card}\} \quad A = \{\text{card is ace}\} \quad \mathrm{P}(F) = \frac{12}{52},\ \mathrm{P}(A) = \frac{4}{52}$$

$$\therefore \qquad \mathrm{P}(F \cup A) = \mathrm{P}(F) + \mathrm{P}(A) - \mathrm{P}(F \cap A) = \frac{12}{52} + \frac{4}{52} - 0 = \frac{16}{52}.$$

2.

$$D = \{\text{double is thrown}\} \qquad N = \{\text{sum is } 9\}$$

$\mathrm{P}(D) = \frac{6}{36}36$

$$\mathrm{P}(N) = \mathrm{P}\{(6 \cap 3) \cup (5 \cap 4) \cup (4 \cap 5) \cup (3 \cap 6)\} = \frac{4}{36}$$

$$\mathrm{P}(D \cup N) = \mathrm{P}(D) + \mathrm{P}(N) - \mathrm{P}(D \cap N) = \frac{6}{36} + \frac{4}{36} - 0 = \frac{10}{36}$$

$$\mathrm{P}((D \cup N)') = 1 - \mathrm{P}(D \cup N) = 1 - \frac{10}{36} = \frac{26}{36}.$$

Checkpoint 3.2.30 This question contains interactive elements.

Colophon

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